Phase Diagrams for Autonomous Differential Equations

In a Nut Shell: First order differential equations of the form

$$dy/dx = f(y)$$

where the function on the right hand side does not depend on the independent variable, x, are called **"autonomous"** and are amenable to separation of variables. Solutions for y(x) may be quite complicated depending on the expression for f(y).

Critical points, y_o, used to identify stability occur where $f(y_o) = 0$. Plots on the real axis line, y, showing the critical points along with the slopes between the critical points are termed **phase diagrams**. They are useful in helping construct slope fields used to identify where the response, y(x), is stable, unstable, or semi-stable.

Strategy for construction of the Phase Diagram

Step 1: Find the critical points. i.e. where the slope is zero. i.e. $f(y) = 0$

Step 2: Construct a real axis, y, and mark the values of y where $f(y) = 0$

Step 3: Find the slopes on either side of the critical points.

Step 4: At each critical point, draw arrows indicating increasing values of slope,
i.e. For positive slope use an arrow pointing in the + y direction.

i.e. For negative slope use an arrow pointing in the **-** y direction.

If the arrows at the critical point are towards each other, then the critical point is stable (i.e.sink).

If the arrows point at the critical point are away from each other, then the critical point is unstable (source)

If the arrows are both in the same direction, then the critical point is semi-stable.

The autonomous differential equation: $dy/dx = y^2 (y^2 - 4)$

Has critical points at $y = 0$ and $y = \pm 2$. i.e. $f(y) = 0$

Example: Find the critical points then sketch the phase diagram for the following differential equation:

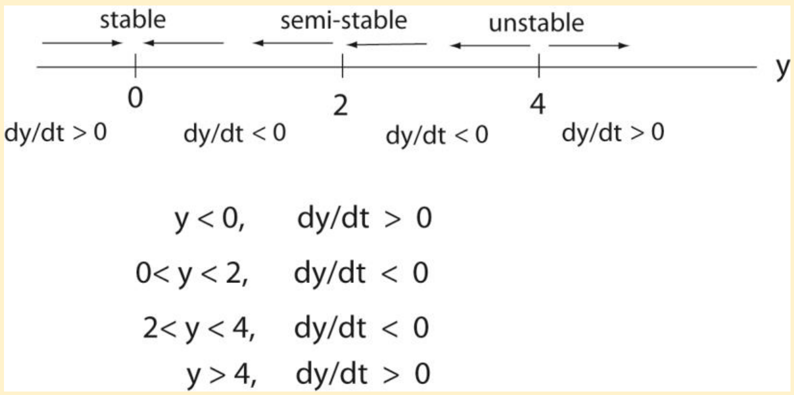
$$dy/dt = (1/2)\, y (y - 2)^2 (y - 4)$$

Strategy: Step 1: Find the critical points by setting $f(y) = 0$.
Critical Points: $f(y) = 0$ at $y = 0$, $y = 2$, and $y = 4$

Step 2: Identify each region where slopes may change from positive to negative, stay the same, or change from negative to positive. Evaluate the slopes on each side of the critical points by taking values with each region.

The four regions in this example include: $y < 0$, $0 < y < 2$, $2 < y < 4$, $y > 4$
i.e. For $y = -1$, $dy/dt = 45/2 > 0$, for $y = 1$, $dy/dt = -3/2 < 0$
For $y = 3$, $dy/dt = -3/2 < 0$, and for $y = 5$, $dy/dt = 45/2 > 0$

Plot Phase Diagram:



Note: A right directed arrow indicates that the dependent variable, y, is increasing.
A left directed arrow indicates that the dependent variable, y, is decreasing.

Results: At the critical point $y = 0$, the response is stable.

At the critical point $y = 2$, the response is semi-stable.

At the critical point $y = 4$, the response is unstable.

i.e. In the neighborhood of $y = 0$, the value of $y(t)$ tends toward a stable value.
In the neighborhood of $y = 2$, the value of $y(t)$ hovers around a stable value.
In the neighborhood of $y = 4$, the value of $y(t)$ departs from a stable value.

Example: Plot the slope field for: $dy/dt = (1/2)\, y\, (y - 2)^2 (y - 4) = f(y)$
Strategy: Set up a table including values of y in each region and calculate the slopes, f(y).

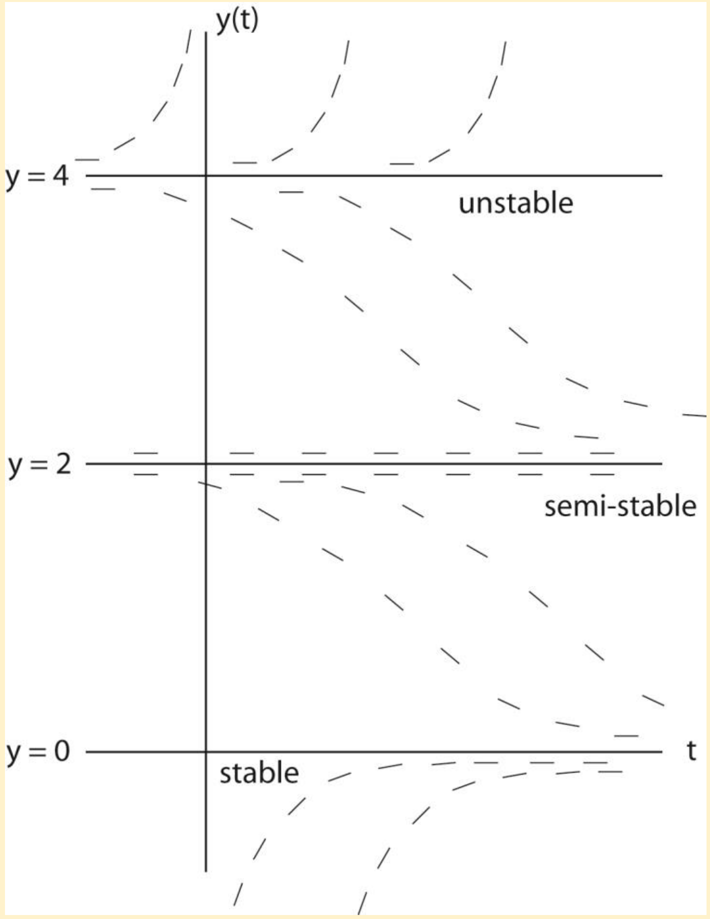
Note: Values of slope are independent on t. So **"solution curves"** are the same for translations in t.

For $y > 4$ $f(y)$ for 4.25, 4.5, 4.75, 5.0, etc   calculate slope and plot

For $2 < y < 4$ $f(y)$ for 2.25, 2.5, 3.0, 3.5, 3.75, etc   calculate slope and plot

For $0 < y < 2$ $f(y)$ for 1.9, 1.75, 1.5, 1.0, 0.5, etc   calculate slope and plot

For $y < 0$ $f(y)$ for -3.0, -2.5, -2.0, -1.5, -0.5 etc   calculate slope and plot



Plot of Slope Field Showing Solution Curves

Qualitative response for y(t) showing solution curves for stable, semi-stable, and unstable regions.