Evaluation of Improper Integrals

In a Nut Shell: Integrals involving unbounded regions are termed improper integrals. The integral may be unbounded by having its limits of integration unbounded (i.e. the upper limit, the lower limit, or both limits of integration unbounded), by having the integrand unbounded, or by having both the limits of integration and the integrand unbounded. Reference these three cases as Type 1, Type 2, and Type 3.

Strategy: Identify where the integral is unbounded, replace the bound with a variable, and then evaluate the integral as the variable approaches its limit.

Type 1: Limits of integration are unbounded; can involve $\pm\infty$

Example: $I = \int_2^\infty [1/x\sqrt{x}]dx \qquad I = \lim_{t\to\infty} \int_2^t [1/x\sqrt{x}]dx$

Here the upper limit is unbounded ($+\infty$) .

Type 2: Integrand is unbounded

Example: $I = \int_0^4 [1/x\sqrt{x}]dx \qquad I = \lim_{t\to 0} \int_t^4 [1/x\sqrt{x}]dx$

Here the integrand is unbounded at $x = 0$.

Type 3: Both the limits of integration (in the example below the upper limit) and the integrand are unbounded (at x = 0).

Example: $I = \int_0^\infty [1/(x + x^2)]dx$

Note both the limits of integration and the integrand are unbounded.

Here upper limit is unbounded ($+\infty$) and also the integrand is unbounded at $x = 0$.

In a situation like this one needs to break region of integration into two parts. Call the integrals I_1 and I_2.

$I_1 = \int_0^1 f(x)\,dx$ and $I_1 = \lim_{t\to 0} \int_t^1 f(x)\,dx$

$$I_1 = \int_0^1 [1/(x+x^2)]dx \; ; \qquad I_1 = \lim_{t\to 0} \int_t^1 [1/(x+x^2)]dx$$

Here the upper limit is arbitrary but select a convenient value (in this case 1).
Pick the variable, t, for the lower limit since the integrand is unbounded at x = 0.

$$I_2 = \int_1^\infty [1/(x+x^2)]dx \; ; \qquad I_2 = \lim_{t\to\infty} \int_1^t [1/(x+x^2)]dx$$

Retain the lower limit of 1 in this second integral (since 1 was selected for the first integral) and pick the variable, t, for the upper limit in the second integral since the upper limit of integration in this integral is unbounded.

The original integral now becomes the sum of two integrals.

$$I = I_1 + I_2$$

Note: If either integral diverges (no limit) then the original integral diverges. So if you evaluate either integral and find that it diverges, then there is no need to evaluate the other integral. The original integral diverges.

Type 1: Limits of integration are unbounded; involve $\pm\infty$

Example: $I = \int_2^\infty (1/x\sqrt{(x)})dx \qquad I = \lim_{t\to\infty} \int_2^t (1/(x\sqrt{x})dx$

Here the upper limit is $+\infty$ so replace the upper limit with the variable t.

$$I = \int_2^\infty (x^{-3/2} dx , \qquad I = \lim_{t\to\infty} \int_2^t (x^{-3/2} dx \quad \text{or} \quad \lim_{t\to\infty} -2x^{-1/2} \Big|_2^t$$

$$I = \lim_{t\to\infty} -2t^{-1/2} + 2(2^{-1/2}) = \sqrt{2}$$ So the integral converges to the value $\sqrt{2}$.

Type 2: Integrand is unbounded

Example: $I = \int_0^4 (1/x\sqrt{(x)})dx \qquad I = \lim_{t\to 0} \int_t^4 (1/(x\sqrt{x})dx$

Here integrand is unbounded at x = 0 . So replace the lower limit with the variable t.

$$I = \int_0^4 (x^{-3/2} dx , \qquad I = \lim_{t\to 0} \int_t^4 (x^{-3/2} dx \quad \text{or} \quad \lim_{t\to 0} -2x^{-1/2} \Big|_t^4$$

$$I = -2(4^{-1/2}) + 2\lim_{t\to 0} x^{-1/2}$$ limit does not exist; This integral diverges.

Type 3: Both the limits of integration and the integrand are unbounded

Example: $I = \int_0^{\infty} [1/(x+x^2)]dx$

Here upper limit is $+\infty$ and integrand is unbounded at $x = 0$.

Here we need to break region of integration into two parts that covers the full range of integration. In so doing pick a convenient upper limit for the first integral such as 1 and continue with the second integral from 1 to ∞ remembering to select a variable for the lower limit of the first integral and a different variable for the upper limit of the second integral to avoid any confusion. i.e.

$I_1 = \int_t^1$ and $I_2 = \int_1^s$ Continue.

$$I_1 = \int_0^1 [1/(x+x^2)]dx \;; \qquad I_1 = \lim_{t \to 0} \int_t^1 [1/(x+x^2)]dx$$

$$I_2 = \int_1^{\infty} [1/(x+x^2)]dx \;; \qquad I_2 = \lim_{s \to \infty} \int_1^s [1/(x+x^2)]dx$$

$$I = I_1 + I_2$$

Note: If either integral diverges (no limit) then the original integral diverges.

In both integrals we can factor the integrand as $1/[x(1+x)]$. So we will **use partial fractions** to evaluate both integrals. i.e. $A/x + B/(1+x)$

Next calculate the values of A and of B as follows:

$A(1+x) + Bx = 1$ so $A = 1$ and $B = -1$

$$\text{So } I_1 = \lim_{t \to 0} \ln|x| - \ln|1+x| \Big|_t^1 = \ln|1| - \ln|2| - \lim_{t \to 0} \ln|t| + \ln|1+t|$$

But $\lim_{t \to 0} \ln|t|$ does not exist. So the integral diverges.

There is no need to check the second integral.

Therefore the original integral $I = \int_0^{\infty} [1/(x+x^2)]dx$ diverges. (result)