

Integrals Involving Quadratic Polynomials in the denominator

In general a polynomial, $P(x)$, of degree n has the form:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_2 x^2 + a_1 x + a_0$$

In a Nut Shell: A quadratic polynomial has the form $ax^2 + bx + c$.

One strategy to evaluate integrals involving quadratic polynomials is to complete the square followed by one or more appropriate trigonometric substitutions as illustrated by the example shown below.

Example: $I = \int dx / (x^2 + 2x + 5)$

Strategy: First complete the square with $(x + 1)^2 + 4 = (x + 1)^2 + 2^2$

$$I = \int dx / [(x + 1)^2 + 2^2] \quad \text{Now let } u = x + 1, du = dx$$

$$I = \int du / (u^2 + 2^2) \text{ which suggests the following trig substitution}$$

(Refer to the section on the application of trig substitutions.)

$$\text{Let } u = 2 \tan \theta, du = 2 \sec^2 \theta d\theta, u^2 + 2^2 = 2^2 \sec^2 \theta$$

$$I = \int 2 \sec^2 \theta d\theta / 2^2 \sec^2 \theta = \int (1/2) d\theta = (1/2) \theta + C$$

$$\text{Now } \theta = \tan^{-1}(u/2) = \tan^{-1}[(x+1)/2]$$

$$\text{Recall } u = x + 1$$

So the integral becomes:

$$I = (1/2) \tan^{-1}[(x+1)/2] + C \quad (\text{result})$$

Note: As a check, differentiation of the result for your integral should return the original integrand if the integration was correct. **Here's the check for the above example.**

$$dI/dx = (1/2) [1 / \{(1 + (x+1)^2/2^2)\}(1/2)] = 1 / [x^2 + 2x + 1 + 4]$$

$$dI/dx = 1 / (x^2 + 2x + 5) \quad \text{check!}$$

In a Nut Shell: For more complicated cases involving quadratic polynomials it may be necessary to split the integral into several integrals. The examples below illustrate this strategy.

Example: $I = \int (2x + 3)dx / (9x^2 + 6x + 5)$ rewrite integral as follows:

Complete the square in the denominator.

$$I = \int [(2x + 3) / (3x + 1)^2 + 2^2] dx$$

$$\text{Let } w = 3x + 1, \quad x = (1/3)(w-1) \text{ and } 2x = (2/3)(w-1)$$

$$dw = 3dx, \quad dx = (1/3) dw \text{ so } 3x = w - 1 \quad 2x = (2/3)(w-1)$$

$$\text{so } I = \int [(2/3)(w-1) + 3] / (w^2 + 2^2) (1/3) dw \quad \text{and}$$

$$I = \int [(2/9)(w-1) + 1] / (w^2 + 2^2) dw = \int (2/9) w / (w^2 + 2^2) dw + (7/9) \int dw / (w^2 + 2^2)$$

Now split into two integrals $I = I_1 + I_2$ where

$$I_1 = \int [(2/9) w / (w^2 + 2^2)] dw \quad \text{and} \quad I_2 = (7/9) \int dw / (w^2 + 2^2)$$

For the **first integral** let $v = w^2 + 2^2$ then $dv = 2w dw$ and $w dw = (1/2) dv$

The integral becomes $I_1 = (2/9) \int (1/2) dv/v$ which is a standard integral

$$I_1 = (1/9) \ln v + C \quad \text{Recall } w = 3x + 1$$

$$I_1 = (1/9) \ln (w^2 + 2^2) + C = (1/9) \ln (9x^2 + 6x + 1 + 4) + C$$

$$I_1 = (1/9) \ln (w^2 + 2^2) + C = (1/9) \ln (9x^2 + 6x + 5) + C$$

For the second integral $I_2 = (7/9) \int dw / (w^2 + 2^2)$

$$\text{Let } w = 2 \tan \theta \quad \text{and} \quad dw = 2 \sec^2 \theta d\theta$$

$$w^2 + 2^2 = 4(1 + \tan^2 \theta) = 4 \sec^2 \theta$$

$$I_2 = (7/9) \int 2 \sec^2 \theta d\theta / 4 \sec^2 \theta = (7/18) \int d\theta = (7/18) \theta + C$$

$$\text{But } w = 2 \tan \theta \quad \text{so } \theta = \tan^{-1}(w/2) = \tan^{-1}[(3x+1)/2]$$

$$\text{And } I = I_1 + I_2$$

$$I = 1/9 \ln |9x^2 + 6x + 5| + (7/18) \tan^{-1}[(3x+1)/2] + C \quad (\text{result})$$

An alternate strategy to evaluate integrals involving quadratic polynomials in the denominator of the integral is to recognize at the outset that the given integral has the form of

$$I = \int du / u$$

$$\text{In the above example } u = 9x^2 + 6x + 5 \text{ and } du = (18x + 6) dx$$

Alternate Solution: * Courtesy of Prof. James W. Phillips

Example: $I = \int (2x + 3)dx / (9x^2 + 6x + 5)$ rewrite integral as follows:

Use the substitution $u = 9x^2 + 6x + 5$ and $du = (18x + 6) dx$

But the term in the numerator is $2x + 3$ not $18x + 6$

So introduce constants A and B such that $2x + 3 = A(18x + 6) + B$ and solve for A and B .

$2x + 3 = A(18x) + 6A + B$ Equate like terms on each side of the equal sign.

$A = 1/9$ $B = 7/3$ So $2x + 3 = (1/9)(18x + 6) + (7/3)$

and $I = \int [(1/9)(18x + 6) + (7/3)] / (9x^2 + 6x + 5) dx$

Now split into two integrals $I = I_1 + I_2$ where

$I_1 = \int (1/9) [(18x + 6) / (9x^2 + 6x + 5)] dx$ and $I_2 = (7/3) \int dx / (9x^2 + 6x + 5)$

For the **first integral** I_1 is of the form: $\int dv/v$ where $v = 9x^2 + 6x + 5$

The integral becomes $I_1 = (1/9) \int dv/v$ which is a standard integral

$$I_1 = (1/9) \ln | 9x^2 + 6x + 5 | + C$$

For the second integral, complete the square in the denominator

$$I_2 = (7/3) \int dx / [(3x+1)^2 + 2^2]$$

Let $w = 3x + 1$ so $dw = 3 dx$ and $dx = (1/3) dw$

$$I_2 = (7/9) \int dw / (w^2 + 2^2)$$

Use the trig substitution $w = 2 \tan \theta$ and $dw = 2 \sec^2 \theta d\theta$

$$w^2 + 2^2 = 4 (1 + \tan^2 \theta) = 4 \sec^2 \theta$$

$$I_2 = (7/9) \int 2 \sec^2 \theta d\theta / 4 \sec^2 \theta = (7/18) \int d\theta = (7/18) \theta + C$$

But $w = 2 \tan \theta$ so $\theta = \tan^{-1} (w/2) = \tan^{-1} [(3x + 1)/2]$

And $I = I_1 + I_2$

$$I = 1/9 \ln | 9x^2 + 6x + 5 | + (7/18) \tan^{-1} [(3x+1)/2] + C \quad (\text{result})$$

Note: Both methods yield the same result for the integral.