

More Integrals involving Trig Functions

In a Nut Shell: General guidelines for integrals involving products of tangent and of secant depend on whether the powers of tangent and secant are even or odd. Here are guidelines combinations of m odd, n even and for m even and n odd.

With m an odd integer and n an even integer; i.e.

Example $I_1 = \int \tan^m x \sec^n x dx$

$$I_1 = \int \tan^3 x \sec^2 x dx \quad \text{Now let } u = \tan x \quad du = \sec^2 x dx$$

$$\text{So } I_1 = \int u^3 du \quad \text{which is a standard integral.}$$

Note: Similar reasoning for the integral: $I_{1a} = \int \cot^m x \csc^n x dx$

Where let $u = \cot x, \quad du = -\csc^2 x dx$

With m an even integer and n an odd integer; i.e.

Example (much harder) $I_2 = \int \tan^m x \sec^n x dx$

$$I_2 = \int \tan^2 x \sec^3 x dx$$

$$\tan^2 x = \sec^2 x - 1$$

$$I_2 = \int (\sec^2 x - 1) \sec^3 x dx$$

$$I_2 = \int (\sec^5 x - \sec^3 x) dx$$

Next use integration by parts on each of these 2 integrals

$$\text{First } I_{2a} = \int \sec^3 x dx$$

$$\text{Next let } u = \sec x \quad dv = \sec^2 x dx$$

$$du = \sec x \tan x dx \quad v = \tan x$$

$$I_{2a} = \int \sec^3 x dx = \sec x \tan x - \int \sec x \tan^2 x dx$$

$$\tan^2 x = \sec^2 x - 1$$

$$-\int \sec x \tan^2 x dx = -\int \sec x (\sec^2 x - 1) dx = -\int \sec^3 x - \sec x dx$$

$$I_{2a} = \int \sec^3 x dx = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$$

$$2 \int \sec^3 x dx = \sec x \tan x + \int \sec x dx \quad \text{or}$$

$$\int \sec^3 x dx = (\sec x \tan x)/2 + (\ln |\sec x + \tan x|)/2 + C$$

A similar strategy applies to $I_{2b} = \int \sec^5 x \, dx$

In this case

$$\begin{aligned} u &= \sec^3 x & dv &= \sec^2 x \, dx \\ du &= 3 \sec^3 x \tan x \, dx & v &= \tan x \end{aligned}$$

$$\text{So } \int \sec^5 x \, dx = \sec^3 x \tan x - 3 \int \sec^3 x \tan^2 x \, dx$$

$$\text{Now } 3 \int \sec^3 x \tan^2 x \, dx = 3 \int \sec^5 x \, dx - 3 \int \sec^3 x \, dx$$

$$\text{Or } \int \sec^5 x \, dx = \sec^3 x \tan x - 3 \int \sec^5 x \, dx + 3 \int \sec^3 x \, dx$$

$$4 \int \sec^5 x \, dx = \sec^3 x \tan x + 3 \int \sec^3 x \, dx$$

$$\text{So } \int \sec^5 x \, dx = (\sec^3 x \tan x)/4 + (3/4) \int \sec^3 x \, dx$$

Another Example $I = \int \tan^3 x \sec^3 x \, dx$

$$I = \int \tan x \tan^2 x \sec^3 x \, dx \quad \text{Use } \tan^2 x = \sec^2 x - 1 \text{ and rearrange,}$$

$$I = \int (\sec^2 x - 1) \sec^2 x (\tan x \sec x) \, dx$$

$$I = \int \sec^4 x - \sec^2 x \tan x \sec x \, dx$$

$$I = \int (\sec^4 x - \sec^2 x) d(\sec x)$$

$$I = (\sec^5 x)/5 - (\sec^3 x)/3 + C$$