Quadratic Surfaces -- Rectangular, Cylindrical and Spherical Coordinates

In a Nut Shell: Besides planes in three dimensions there exist a family of cylinders and quadric surfaces that are of interest in mathematics. A quadric surface is the graph of a second-degree equation in three variables x, y, and z. You may be asked to identify and draw figures of these cylindrical and quadric surfaces.

Definition of Cylinders A cylinder is a surface that consists of all lines (called rulings) that are parallel to a given line and pass through a given plane curve.

Three examples include:

$x^2 + y^2 = a$	a cylinder with its axis along the z-axis of radius a
$y^2 + z^2 = a$	a cylinder with its axis along the x-axis of radius a
$z = x^2$	a parabolic cylinder with rulings parallel to the y-axis

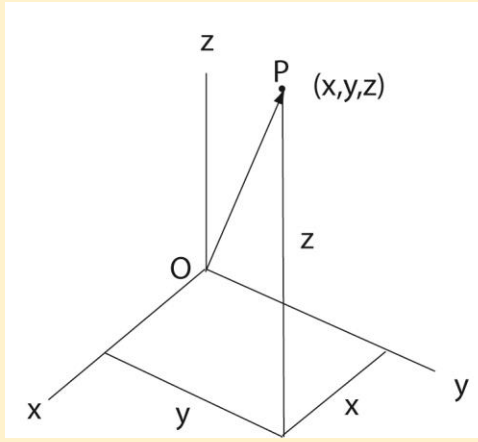
The **family of quadric surfaces** include the ellipsoid, the elliptic paraboloid, the elliptic cone, the hyperboloid of one sheet, the hyperboloid of two sheets, and the hyperbolic paraboloid. The table below gives expressions for each quadric surface.

Table of Quadric Surfaces (Current software lets you plot these surfaces.)

$(x/a)^2 + (y/b)^2 + (z/c)^2 = 1$	ellipsoid
$(x/a)^2 + (y/b)^2 = (z/c)$	elliptic paraboloid
$(x/a)^2 + (y/b)^2 = (z/c)^2$	elliptical cone
$(x/a)^2 + (y/b)^2 - (z/c)^2 = 1$	hyperboloid of one sheet
$(z/c)^2 - (x/a)^2 - (y/b)^2 = 1$	hyperboloid of two sheets
$(y/b)^2 - (x/a)^2 = z/c \quad c > 0$	hyperbolic paraboloid

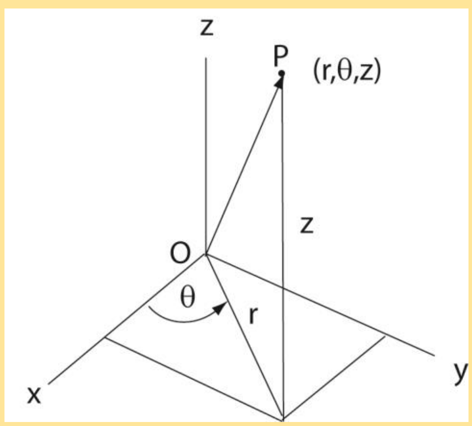
In a Nut Shell: Three common coordinate systems used in vector calculus include: **Rectangular** (x, y, z) coordinates, **Cylindrical** (r , θ, z) coordinates, and **Spherical** (ρ, θ, φ) coordinates. You should be able to use each of these coordinate systems and also be able to transform them from one to another.

Rectangular Coordinates of a point, P, in space: (x, y, z)



Cylindrical Coordinates of a point in space: (r , θ, z)

where θ = angle between the x-axis and the radius, r, in the x-y plane



Spherical Coordinates of a point in space:

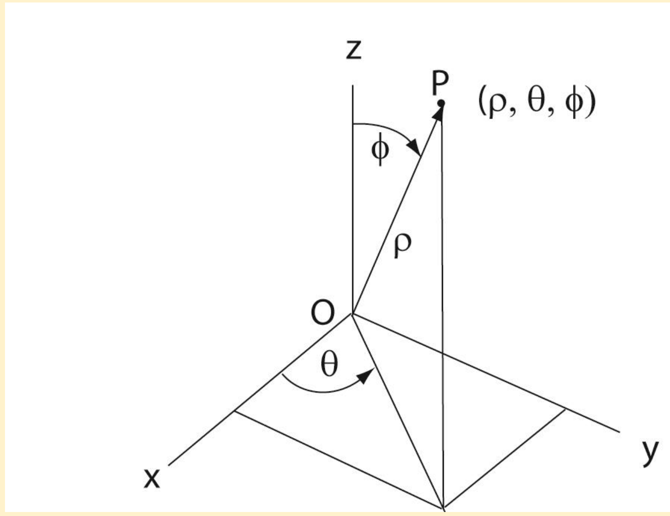
Let ρ = magnitde of vector from the origin, O, out to the point P

θ = angle between the x-axis and the line formed by the projection of ρ on to the x-y plane

φ = angle between the z-axis and ρ

So the spherical coordinate of a point P in space is given by:

$$(\rho \sin \varphi \cos \theta , \ \rho \sin \varphi \sin \theta , \ \rho \cos \varphi)$$



Transformation from rectangular to cylindrical:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$ **Note that:** $x^2 + y^2 + z^2 = r^2$

Transformation from rectangular to Spherical:

$x = \rho\sin\varphi\cos\theta$, $y = \rho\sin\varphi\sin\theta$, $z = \rho\cos\varphi$

Note that: $x^2 + y^2 + z^2 = \rho^2$

Transformation from cylindrical to spherical:

$r = \rho\sin\varphi$ $x^2 + y^2 = r^2 = \rho^2(\sin\varphi)^2$ $z = \rho\cos\varphi$

Example: Convert the equation below from rectangular to cylindrical coordinates.

$$x^2 + y^2 + z^2 = x + y + z$$

Strategy: Substitute: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

So $(r\cos\theta)^2 + (r\sin\theta)^2 + (z)^2 = r\cos\theta + r\sin\theta + z$

$$r^2[\cos^2\theta + \sin^2\theta] + z^2 = r\cos\theta + r\sin\theta + z$$

$$r^2 + z^2 = r\cos\theta + r\sin\theta + z \quad \text{(result)}$$

Example: Convert the equation below from rectangular to spherical coordinates.

$$x^2 + y^2 + z^2 = x + y + z$$

Strategy: Substitute: $x = \rho \sin \varphi \cos \theta$, $y = \rho \sin \varphi \sin \theta$, $z = \rho \cos \varphi$

$$(\rho \sin \varphi \cos \theta)^2 + (\rho \sin \varphi \sin \theta)^2 + (\rho \cos \varphi)^2 =$$

$$\rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta + \rho \cos \varphi$$

$$(\rho \sin \varphi)^2 [\cos^2 \theta + \sin^2 \theta] + \rho \cos^2 \varphi = \rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta + \rho \cos \varphi$$

$$\rho^2 (\sin^2 \varphi + \cos^2 \varphi) = \rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta + \rho \cos \varphi$$

$$\rho^2 = \rho \sin \varphi \cos \theta + \rho \sin \varphi \sin \theta + \rho \cos \varphi$$

or

$$\rho = \sin \varphi \cos \theta + \sin \varphi \sin \theta + \cos \varphi$$

$$\rho = \sin \varphi (\cos \theta + \sin \theta) + \cos \varphi \qquad \text{(result)}$$