Sturm-Liouville Applications

In a Nut Shell: The Sturm-Liouville Problem involves solving a differential equation having special properties along with associated boundary conditions defined in the table below.

The Sturm-Liouville D.E. and associated boundary conditions:

$$\frac{d}{dx}\left[p(x) \frac{dy}{dx}\right] - q(x)\, y + \lambda\, r(x)\, y = 0 \quad \text{where } a < x < b$$

$$A_1\, y(a) - A_2\, y'(a) = 0 \qquad \text{and} \qquad B_1\, y(b) + B_2\, y'(b) = 0$$

where neither A_1 and A_2 nor B_1 and B_2 are both zero.

The objective is to find the eigenvalues, λ, that yield solutions to the differential equation satisfying the prescribed boundary conditions.

Sturm-Liouville Problems have several useful properties that apply to their solution. The table below lists these properties.

Property 1: If $p(x)$, dp/dx, $q(x)$, and $r(x)$ are continuous in [a,b] and if $p(x) > 0$ ad $r(x) > 0$, then the eigenvalues are nonnegative.

Property 2: Eigenfunctions of the Sturm-Liouville problem are orthogonal on the interval with respect to the weight function $r(x)$.

[a,b] . i.e. if $y_n(x)$ and $y_m(x)$ are eigenfunctions, then

$$\int_a^b y_n(x)\, y_m(x)\, r(x)\, dx = 0 \quad \text{for} \quad m \neq n$$

Property 3: A function $f(x)$ can be represented in the interval [a,b] by an eigenfunction series. i.e.

$$f(x) = \sum_{n=1}^{\infty} c_n\, y_n(x)$$

Example: Find the eigenvalues and eigenfunctions for the following problem:

$$y'' + \lambda y = 0 \qquad \text{here } p(x) = 1,\ q(x) = 0, \text{ and } r(x) = 1$$

$$hy(0) - y'(0) = 0, \quad h > 0 \qquad \text{and} \qquad y(L) = 0$$

Note: Since both $p(x)$ and $r(x) > 0$ there are no negative eigenvalues. So the only Possible eigenvalues are $\lambda = 0$ and $\lambda > 0$. The table below contains these two cases.

Case 1: $\lambda = 0$ $y(x) = Ax + B$, $y' = A$

$hy(0) - y'(0) = 0 = hB - A = 0$, $A = hB$

$y(L) = 0 = AL + B = hB + B = 0 = (h + 1)B = 0$

Since $h > 0$, $B = 0$ and $A = hB = 0$

So there are no eigenvalues nor eigenfunctions for $\lambda = 0$.

Case 2: $\lambda > 0$, $\lambda = \alpha^2$ $y'' + \alpha^2 y = 0$

$y = A \cos \alpha x + B \sin \alpha x$ and $y' = - A\alpha \sin \alpha x + B\alpha \cos \alpha x$

$hy(0) - y'(0) = 0 = hA - B\alpha$, $B = hA/\alpha$

So $y = A \cos \alpha x + (hA/\alpha) \sin \alpha x = A [\cos \alpha x + (h/\alpha)\sin \alpha x]$ --------- (1)

and $y(L) = 0 = A [\cos \alpha L + (h/\alpha)\sin \alpha L]$ for a nontrivial solution $A \neq 0$

Let $\beta_n = \alpha_n L$ so $\alpha_n = \beta_n/L$ -------------------------- (2)

So $\cos \beta_n + (hL/ \beta_n)\sin \beta_n] = 0$

$\tan \beta_n = - \beta_n /hL$ (transcendental equation for roots)

where the **eigenvalues are** $\lambda_n = \alpha_n^2 = (\beta_n/L)^2$

and from (1) with (2) the **eigenfunctions are** $y_n(x) = \beta_n \cos \beta_n x/L + hL\sin \beta_n x/L$

where β_n is the nth positive root of $\tan x = - x/hL$ (hint: use a plot)