Integrals using Trig Substitutions

In a Nut Shell: Integrals, $\int f(x)\,dx$, where the integrand, $f(x)$, has one of the following forms (note: the exponent can be to any power, p)

Case a: $f(x) = (a^2 - x^2)^p$ where a is a constant
Case b: $f(x) = (a^2 + x^2)^p$ where a is a constant
Case c: $f(x) = (x^2 - a^2)^p$ where a is a constant

have convenient trig substitutions that lead to the evaluation of the integral.

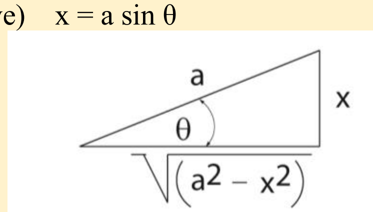
For integrals involving $f(x) = (a^2 - x^2)^p$ the trig substitution $x = a \sin\theta$ is helpful. In this case $dx = a \cos\theta$ and $(a^2 - x^2) = a^2 \cos^2\theta$.

For integrals involving $f(x) = (a^2 + x^2)^p$ the trig substitution $x = a \tan\theta$ is helpful. In this case $dx = a \sec^2\theta\, d\theta$ and $(a^2 + x^2) = a^2 \sec^2\theta$.

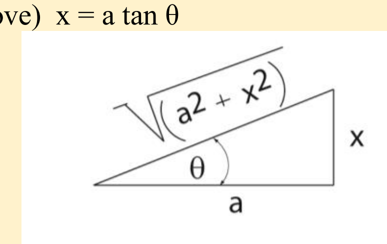
For integrals involving $f(x) = (x^2 - a^2)^p$ the trig substitution $x = a \sec\theta$ is helpful. In this case $dx = a \sec\theta \tan\theta\, d\theta$ and $(x^2 + a^2) = a^2 \tan^2\theta$.

The procedure is to make the substitution, then evaluate the integral in terms of the new variable, θ, and finally convert back into the original variable, x. This conversion is best completed using the following diagrams.

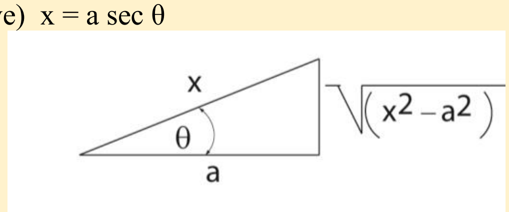
For case a (above) $x = a \sin\theta$



For case b (above) $x = a \tan\theta$



For case c (above) $x = a \sec\theta$



Example involving $(x^2 - a^2)^p$

Evaluate the integral $I = \int dx / [(ax)^2 - b^2]^{3/2}$ (here $p = 3/2$)

First let $u = ax$, then $du = a\,dx$ and $dx = (1/a)\,du$ (u is an intermediate variable)

The integral becomes $I = (1/a) \int du / [u^2 - b^2]^{3/2}$

In this case use the "trig substitution" $u = \sec\theta$, so $du = \sec\theta \tan\theta\, d\theta$

and $[u^2 - b^2] = b^2 \sec^2\theta - b^2 = b^2 \tan^2\theta$ so

$I = (1/a) \int [(b \sec\theta \tan\theta\, d\theta) / (b^3 \tan^3\theta)]\, d\theta = [(1/a\, b^2) \int \sec\theta / \tan^2\theta]\, d\theta$

or $I = (1/a\, b^2) \int \cos^2\theta / \sin^2\theta \cos\theta]\, d\theta = (1/a\, b^2) \int \cos\theta / \sin^2\theta]\, d\theta$

Now let $w = \sin\theta$ $dw = \cos\theta$ and the integral becomes

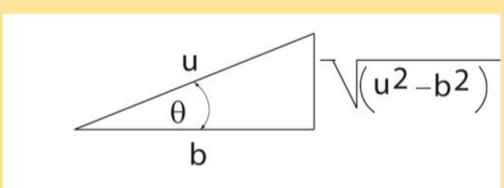
$I = (1/a\, b^2) \int dw / w^2 = (1/a\, b^2) (-1/w) + C = (- 1 /a\, b^2) (1/\sin\theta) + C$

or $I = (- 1/a\, b^2) \csc\theta + C$

Now express the result in terms of the original variable x.

Start with the triangle involving the intermediate variable, u. Recall $u = \sec\theta$.

The relevant triangle is:



Therefore $\sec\theta = u / b$ and $\csc\theta = u / \sqrt{(u^2 - b^2)}$ and also recall $u = ax$

So $I = (- 1/a\, b^2)\ ax / \sqrt{[(ax)^2 - b^2]} + C$ or finally

$I = (- 1/ b^2) \{ x / \sqrt{[(ax)^2 - b^2]} \} + C$ (result)

Example involving $(x^2 + a^2)^p$

Evaluate the integral $I = \int x^5\, dx / \sqrt{(x^2 + 2)}$ (here $p = 1/2$)

First let $x = \sqrt{2} \tan\theta$, then $dx = \sqrt{2} \sec^2\theta\, d\theta$ and $x^2 + 2 = 2\tan^2\theta + 2$

or $x^2 + 2 = 2 \sec^2\theta$

The integral becomes $I = \int [\, 2^{5/2} \tan^5\theta\, \sqrt{2} \sec^2\theta\,] / [\, \sqrt{2} \sec\theta\,]\, d\theta$

or the integral becomes $I = 2^{5/2} \int \tan^5\theta\, \sec\theta\, d\theta$

Next write the integral as $2^{5/2} \int \tan^4\theta\, \sec\theta \tan\theta\, d\theta$

And use $\tan^2\theta = \sec^2\theta - 1$ Then the integral becomes

$I = 2^{5/2} \int (\sec^2\theta - 1)^2 \sec\theta \tan\theta\, d\theta$

Next introduce the variable $u = \sec\theta$ so that $du = \sec\theta \tan\theta\, d\theta$

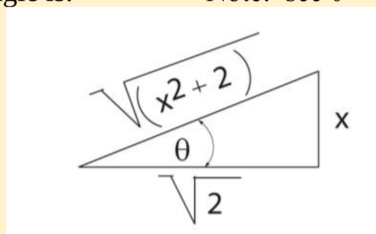
The integral becomes $I = 2^{5/2} \int (u^2 - 1)^2\, du = 2^{5/2} \int (u^4 - 2u^2 + 1)\, du$

$I = 2^{5/2} [\, u^5/5 - 2u^3/3 + u\,] + C$ where $u = \sec\theta$

or $I = 2^{5/2} [\, (1/5)\sec^5\theta - (2/3)\sec^3\theta + \sec\theta + C$

Now express the result in terms of the original variable x. Recall $x = \sqrt{2} \tan\theta$.

The relevant triangle is: Note: $\sec\theta = \sqrt{[\, (x^2 + 2) / \sqrt{2}\,]}$



or $I = 2^{5/2} [\, (1/5) [(x^2 + 2)/2]^{5/2} - (2/3) [(x^2 + 2)/2]^{3/2} + [(x^2 + 2)/2]^{1/2} + C$