Approximate Methods for First Order Differential Equations

In a Nut Shell: Not all first order differential equations (linear or nonlinear)

$$dy/dx = f(x,y)$$

can be solved exactly and explicitly. For example you may not be able to integrate f(x,y) to obtain y(x). In such cases there are three approximate methods that you can use. They include **Picard's Method, Euler's Method,** and the **Runge-Kutta Method**.

Problem Statement: Find an approximate solution to the first order, ordinary d.e.

$$dy/dx = f[(x,y(x)] \quad \text{subject to the condition} \quad y(a) = b$$

Note:

- Approximate methods apply to first order differential equations that may be linear or nonlinear.
- Each numerical method may contain local and cumulative errors.
- You need to decide on an appropriate step size for each method. Larger step sizes get you to the numerical solution quicker but may not be as accurate.

  Smaller step sizes will take longer but each iteration carries with it a local error which may accumulate.

Method 1: Picard's Method

Problem Statement: Find an approximate solution to the first order, ordinary d.e.

$$dy/dx = f[(x,y(x)] \quad \text{subject to the condition} \quad y(a) = b$$

Strategy: Use successive approximations (another name for Picard's Method)

First Integrate $dy = f[x,y(x)]\,dx$ from a to c or $\int_a^c dy = \int_a^c f[x,y(x)]\,dx$

where a is the initial value of x and c is an arbitrary value.

Then $y(c) - y(a) = \int_a^c f[x,y(x)]\,dx$ but $y(a) = b$ so

$$y(c) - b = \int_a^c f[x,y(x)]\,dx$$

here x is a dummy variable (switch variable of integration to t) for arbitrary x, $y(x) = b + \int_a^x f[t, y(t)]\, dt$

Successive steps are given by: $y_{n+1}(x) = b + \int_a^x f[t, y_n(t)]\, dt$ start with $y_o = b$

Example: Use Picard's Method to find the first, second, and third approximations to

$dy/dx = 1 + y^2$, $y(0) = 3$ i.e. $y_1(x)$, $y_2(x)$, and $y_3(x)$

and evaluate estimate at $x = 0.1$. Further compare with the exact solution.

Strategy: Use successive approximations by applying the following:

$$y_{n+1}(x) = b + \int_a^x f[t, y_n(t)]\, dt$$

where $b = 3$ and $a = 0$ and start with $y_o = b = 3$

So $y_1(x) = 3 + \int_0^x [1 + y_o(t)^2]\, dt = 3 + \int_0^x [1 + 3^2]\, dt = 3 + 10x$

Now $y_2(x) = b + \int_a^x f[t, y_1(t)]\, dt = 3 + \int_0^x [1 + (3 + 10t)^2\, dt$

So $y_2(x) = 3 + 10x + 30x^2 + (100/3)\, x^3$

Now $y_3(x) = b + \int_a^x f[t, y_2(t)]\, dt =$

$$= 3 + \int_0^x [1 + (3 + 10t + 30t^2 + (100/3)\, t^3)^2\, dt$$

After integration:

$y_3(x) = 3 + 10x + 30x^2 + (280/3)\, x^3 + 200x^4 +$

$(4700/15)\, x^5 + (2000/6)\, x^6 + (10{,}000/63)\, x^7$

$y(x) = \tan[x + \tan^{-1}(3)]$	$y(0.1) = 4.43541$
$y_1(x) = 3 + 10x$	$y_1(0.1) = 4.00000$
$y_2(x) = 3 + 10x + 30x^2 + (100/3)x^3$	$y_2(0.1) = 4.33333$
$y_3(x) = 3 + 10x + 30x^2 + (280/3)x^3 + 200x^4 + (4700/15)x^5 + (2000/6)x^6 + (10{,}000/63)x^7$	$y_3(0.1) = 4.41648$

Now compare with

the exact solution for the initial value problem

$dy/dx = 1 + y^2$, $y(0) = 3$ **Note:** The d.e. is nonlinear.

Strategy: Use separation of Variables:

Separate variables: $dy / (1 + y^2) = dx$

Integrate: $\int dy / (1 + y^2) = \int dx$ which gives: $\tan^{-1}(y) = x + C$

Or $y = \tan(x + C)$ Use the initial condition $y(0) = 3$ to obtain $C = \tan^{-1}(3)$

The result is: $y(x) = \tan[x + \tan^{-1}(3)]$

Method 2: Euler's Method

In a Nut Shell: Euler's Method provides a simple approach to obtain a numerical solution of linear or nonlinear first order differential equations. However, care must be exercised since convergence is not guaranteed. Numerical errors may accumulate leading to erroneous results.

Problem Statement: Find an approximate solution to the first order, ordinary d.e.

$$dy/dx = f[(x, y(x)] \quad \text{subject to the condition} \quad y(0) = y_o$$

Strategy: Use Euler's Algorithm to obtain successive approximations

$$y_{n+1} = y_n + h\, f(x_n, y_n) \qquad (n \ge 0)$$

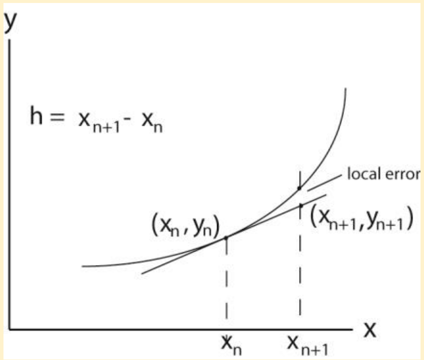
where h = step size

Of course smaller step sizes require more steps to arrive at a result whereas larger step sizes may yield less accurate results.

Local and Cumulative Errors related to Euler's Method.

The linear approximation to the solution curve is as follows:

$y(x_{n+1}) \approx y_n + h\, f(x_n, y_n) = y_{n+1}$ The figure below shows the local error.



Trade-off: The local error can accumulate. So selection of the step size, h, is an Important consideration.

Example: Use Euler's Method twice to find the approximate solution to

$$dy/dx = y + 1 \qquad \text{where } y(0) = 1 \qquad \text{in the interval } [0, 0.5]$$

First use the step size, h, = 0.25 and then use 0.1. Compare your results with the Exact solution $y(x) = 2e^x - 1$ for $x = 0.5$.

Strategy: Apply Euler's Algorithm to obtain successive approximations

$$y_{n+1} = y_n + h\, f(x_n, y_n)$$

Case 1: $h = 0.25$ $f(x, y(x)) = y + 1$, $y_0 = y(0) = 1$

$y_1 = y_0 + h\, f(x_0, y_0) = 1 + 0.25\,[\,1 + 1\,] = 1.5$ (result when x = 0.25)

$y_2 = y_1 + h\, f(x_1, y_1) = 1.5 + 0.25\,[\,1.5 + 1\,] = 2.125$ (result when x = 0.5)

Case 2: $h = 0.1$ $f(x, y(x)) = y + 1$, $y_0 = y(0) = 1$

$y_1 = y_0 + h f(x_0, y_0) = 1 + 0.1 [1 + 1] = 1.2$

$y_2 = y_1 + h f(x_1, y_1) = 1.2 + 0.1 [1.2 + 1] = 1.42$ (result when $x = 0.2$)

$y_3 = y_2 + h f(x_2, y_2) = 1.42 + 0.1 [1.42 + 1] = 1.662$ (result when $x = 0.3$)

$y_4 = y_3 + h f(x_3, y_3) = 1.662 + 0.1 [1.662 + 1] = 1.9282$ (result when $x = 0.4$)

$y_5 = y_4 + h f(x_4, y_4) = 1.9282 + 0.1 [1.9282 + 1] = 2.221$ (result when $x = 0.5$)

Case 3: Exact Solution

$y(x) = 2 e^x - 1$ for $x = 0.5$: $y(0.5) = 2 e^{0.5} - 1 = 2.297$ (result)

Method 3: The Runge-Kutta Method

In a Nut Shell: The Runge-Kutta Method is a "fourth order" method and provides greater accuracy than Picard's or Euler's Methods although it involves more extensive calculations.

Problem Statement: Find an approximate solution to the first order, ordinary d.e.

$dy/dx = f [(x,y(x)]$ subject to the condition $y(0) = y_o$ in $[a,b]$

Strategy: Use the Runge-Kutta Algorithm as given below

$y_{n+1} = y_n + h k$ $n \geq 0$ where h = step size

$k = (1/6) [k_1 + 2k_2 + 2k_3 + k_4]$

and $k_1 = f(x_n, y_n)$, $k_2 = f(x_n + h/2 , y_n + hk_1 / 2)$

$k_3 = f(x_n + h/2 , y_n + hk_2 / 2)$, $k_4 = f(x_{n+1} , y_n + hk_3)$

Example: Use the Runge-Kutta Method to find an approximate solution to

$dy/dx = 2 y$, $y(0) = 0.5$ at $x = 0.5$ using step size, $h = 0.25$

The exact solution is $y(x) = (1/2) e^{2x}$.

Compare the approximate solution with the exact solution at $x = 0.5$.

Strategy: Use the Runge-Kutta Algorithm as given below

$y_{n+1} = y_n + h\,k$ $f(x,y) = 2y$ where h = step size

$k = (1/6)\,[\,k_1 + 2k_2 + 2k_3 + k_4\,]$

and $k_1 = f(x_n, y_n)$, $k_2 = f(\,x_n + h/2\,,\, y_n + hk_1 / 2\,)$

$k_3 = f(\,x_n + h/2\,,\, y_n + hk_2 / 2\,)$, $k_4 = f(\,x_{n+1}\,,\, y_n + hk_3\,)$

$y_1 = y_0 + h\,k$ $y_0 = 0.5$ $f(x,y) = 2y$

$k_1 = f(x_0, y_0) = 2y_0 = 2\,(0.5) = 1.0$

$k_2 = f(\,x_0 + h/2\,,\, y_0 + hk_1 / 2\,) = 2[0.5 + 0.25(1.0/2)] = 1.25$

$k_3 = f(\,x_0 + h/2\,,\, y_0 + hk_2 / 2\,) = 2[0.5 + 0.25(1.25/2)] = 1.3125$

$k_4 = f(\,x_1\,,\, y_0 + hk_3\,) = [0.5 + 0.25(1.3125)] = 1.65625$

So $\sum k_i = 1.0 + 2(1.25) + 2(01.3125) + 1.65625 = 7.78125$

$k = \sum k_i / 6 = 1.296875$

$y_1 = y_0 + h\,k = 0.5 + 0.25[1.296875] = 0.82421875$ at $x = 0.25$

$y_2 = y_1 + h\,k$ $y_1 = 0.82421875$

$k_1 = f(x_1, y_1) = 2y_1 = 2\,(0.82421875) = 1.6484375$

$k_2 = f(\,x_1 + h/2\,,\, y_1 + hk_1 / 2\,) = 2[(0.82421875 + 0.25((0.82421875/2)] = 2.060546875$

$k_3 = f(\,x_1 + h/2\,,\, y_1 + hk_2 / 2\,) = 2[(0.82421875 + 0.25(2.060546875/2)] = 2.163574219$

$k_4 = f(\,x_2\,,\, y_1 + hk_3\,) = 2[(0.82421875 + 0.25(2.163574219)] = 2.73022469$

$k = (1/6)\,[\,k_1 + 2k_2 + 2k_3 + k_4\,]$ $y_{exact} = 2\,e^{2(0.5)} = 1.359140914$

So $k = [1.6484375 + 2(2.060546875) + 2(2.163574219) + 2.73022469]/6 = 2.137817383$

$y_2 = y_1 + h\,k = 0.82421875 + 0.25\,[2.137817383\,] = 1.358673096$ at $x = 0.5$