First Order Nonlinear Differential Equations

In a Nut Shell: There are two forms of first order, nonlinear d.e.'s that appear in elementary differential equations -- those that are separable (easiest type) and those that are nonseparable.

For separable forms, the key step is to place all the dependent variables and their derivatives on one side of the differential equation and all the independent variables on the other side.

The general approach for nonseparable ones is to find a transformation that yields a linear differential equation. The type of transformation depends on the form of the nonlinear d.e.

Start with the easiest type -- first order, nonlinear, d.e.'s that are separable

$$dy/dx \ = \ y' \ = \ f(x,y)$$

Suppose $f(x,y)$ can be expressed as follows: $f(x,y) = g(x)\,h(y)$
Then the d.e. becomes $dy/dx = g(x)\,h(y)$ and separation of variables gives

$dy/h(y) = g(x)\,dx$ which can be integrated directly to obtain $y(x)$.

Nonseparable, first order nonlinear differential equations are more challenging.

The KEY STRATEGY: Use known transformations based on the form of the d.e.

Form of Differential Equation	Transformation
$dy/dx = f(ax + by + c)$	Try $v(x) = ax + by + c$
Homogeneous equations $dy/dx = f(y/x)$	Try $v(x) = y/x$
Bernoulli equations $y' + p(x)\,y = q(x)\,y^n$	Try $v(x) = y^{1-n}$

So the first step is to identify the type of differential equation in order pick the relevant substitution.

Example: Solve the following d.e. $y' = dy/dx = x^2y^3$

Strategy: Note that the Form of D.E. is $dy/dx = f(x,y) = g(x)\,h(y)$

So the dependent variable, y, and the independent variable, x, are separable.

Here $f(x,y) = g(x)\,h(y) = x^2y^3$ (nonlinear because of the term y^3)

So separate the dependent and independent variables using algebra.

$dy/y^3 = x^2\,dx$ Here the variables y and x are separated.

Next integrate once to obtain the solution. **Note:** C is the constant of integration.

$$-\,(½)\,y^{-2} = (1/3)\,x^3 + C$$

$$1/y^2 = -\,(2/3)\,x^3 + C$$

$$y = \sqrt{[\ (1/(C - (2/3)\,x^3)\ ]}$$ (result)

Example: Solve the following d.e. $dy/dx = (9x + y + 10)^2$

Note: The form of D.E. is $dy/dx = f(ax + by + c)$

Strategy: Try the substitution: $v = 9x + y + 10$ then $dv/dx = 9 + dy/dx$

so that $dv/dx - 9 = v^2$ or $dv/dx = 9 + v^2$ **which is separable**

In separable form $dv/(9 + v^2) = dx$ which can be integrated to obtain v.

Hint: Use $v = 3\tan\theta$ to aid in the integration. The result is

$\tan^{-1}(v/3) = 3x + C$ or $v/3 = \tan(3x + C)$

But $v = 9x + y + 10$ so $y = 3\tan(3x + C) - 9x - 10$ (result)

Example: Solve the following d.e. $x^2\,dy/dx = xy + x^2 e^{y/x}$

This nonlinear d.e. is not in standard form. So first divide each term by x^2 to obtain

$$dy/dx = y/x + e^{y/x}$$

Note: The Form of D.E. is $dy/dx = f(y/x)$ "Homogeneous form"

Strategy: Try the transformation $v = y/x$ or $y = vx$ and $dy/dx = x\,dv/dx + v$

The result is $x\,dv/dx + v = v + e^{v}$ or $x\,dv/dx = e^{v}$ which is separable

$e^{-v}\,dv = dx/x$ which after integration gives $-e^{-v} = \ln x + C_1$ or

$e^{-v} = C_2 - \ln(1/x)$

Now $v = y/x$ so $e^{-y/x} = C_2 - \ln(1/x)$ next take the log of both sides

$(-y/x) = \ln[C_2 - \ln x]$ Thus

$y(x) = x \ln[1/(C_2 - \ln x)]$ (result)

Example: Solve the d.e. $2xyy' = 4x^2 + 3y^2$

Next use algebra to put the d.e. into standard form by dividing each term by $2xy$.

$y' = 2x/y + 3/2\,y/x$ or further rearranging to get

$y' - (3/2x)\,y = (2x/y)$ which is in standard Bernoulli form

Form of D.E. is $dy/dx + p(x)\,y = q(x)\,y^n$ "Bernoulli form"

So in this example $p(x) = -(3/2x)$, $q(x) = 2x$, and $n = -1$

Strategy: Try the transformation: $v = y^{1-n}$ where $n = -1$.

which gives $v(x) = y^2$ and differentiating $2yy' = v'$

or $2xyy' = xv'$ so putting this and $v = y^2$ into the original d.e. gives

$xv' = 4x^2 + 3v$ Next rearrange to give

$v' - (3/x)\,v = 4x$ which is a linear d.e.

I.F. = integrating factor = $e^{-\int 3/x)\,dx} = e^{-3\ln x} = x^{-3}$

Multiply each term of the d.e. by x^{-3} to give

$x^{-3}v' - x^{-3}(3/x)\,v = 4/(x^2)$

$d/dx\,[x^{-3}v] = 4/(x^2)$ so $x^{-3}v = -4/x + C$

$x^{-3}y^2 = -4/x + C$, $y^2 = -4x^2 + Cx^3$ (result)