Numerical Integration/Five Cases

In a Nut Shell: Not all functions yield to integration. For example, you may have a set of experimental data for which you would like to perform an integration. In such cases, one may use numerical integration.

Strategy: The general approach is to represent the "area" under the curve as a finite collection of small areas. These areas may be in the shape of small rectangles, small trapezoids, or perhaps small "parabolic bounded" areas under the "curve" of data.

Approximate area under $y = f(x)$ using n small rectangles

$$\text{Area} = \int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i^*)\,\Delta x \qquad \text{where}$$

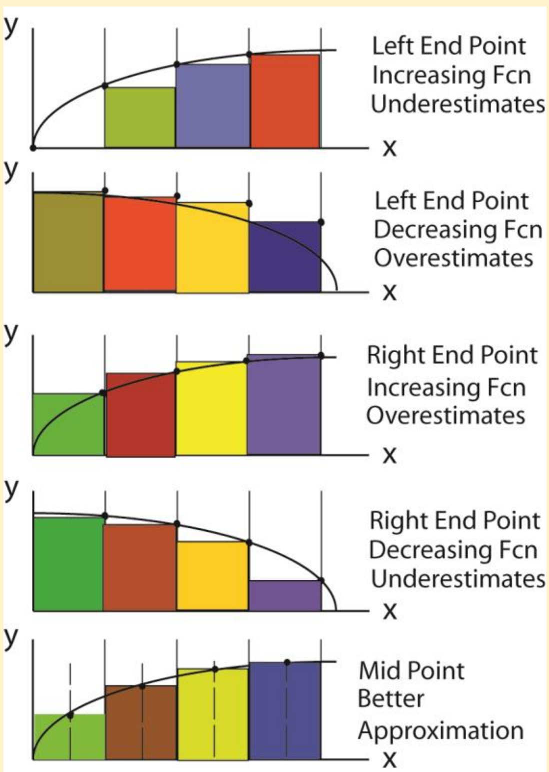
n is the number of subintervals

x_i^* is any point in the ith subinterval (called the sampling point)

Three common sampling points for a rectangle are:

the left endpoint
the midpoint
the right endpoint

The location of the sampling points impacts the estimation of areas for increasing and for decreasing functions as shown in the figures below.



One has five cases for numerical integration of areas. The first three involve the use of rectangles. Case four uses trapezoids and case five uses parabolas. Discussion and examples will be taken in the following order.

Case 1: Procedure and an example using the right endpoint .
Case 2: Procedure and an example using the left endpoint.
Case 3: Procedure and an example using the midpoint.

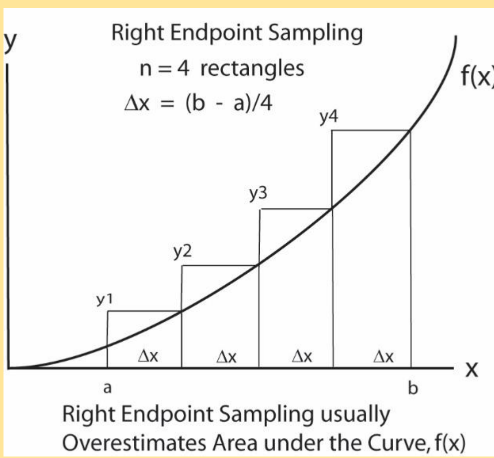
For a better approximation one may break the area under the curve into n trapezoids instead of n rectangles. This approximation leads to Case 4.

Case 4: Procedure and an example using trapezoids.

For an even better approximation one may break the area under the curve into n regions bounded by parabolas instead of n trapezoids. This approximation leads to Case 5.

Case 5: Procedure and an example using parabolas.

Estimate the area under the curve $y(x) = x^2$ from $x = 1$ to $x = 3$ using the right end point.
Note: This function happens to be an "increasing" function (concave up).



Note: The approximate area = $(y_1 + y_2 + y_3 + y_4)\,\Delta x$

Further Note: In this application use of the right end sampling point will result in an overestimation of the actual area.

Here b = 3 a = 1, n = 4 so $\Delta x = (3 - 1)/4 = 0.5$

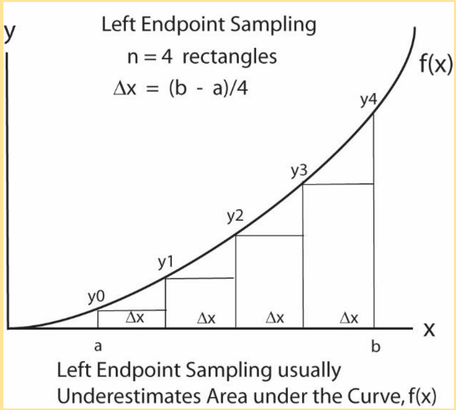
$x_1 = 1.5$, $x_2 = 2$, $x_3 = 2.5$, and $x_4 = 3$

$y_1 = 2.25$, $y_2 = 4$, $y_3 = 6.25$, and $y_4 = 9$

Approximate area = (2.25 + 4 + 6.5 + 9) 0.5 = 10.875

Actual value of area = 8.66666

Estimate the area under the curve $y(x) = x^2$ from $x = 1$ to $x = 3$ using the left end point.
Note: This function happens to be an "increasing" function. (concave up)



Note: The approximate area = (y0 + y1 + y2 + y3) Δx
Further Note: In this application use of the right end sampling point will result in an underestimation of the actual area.

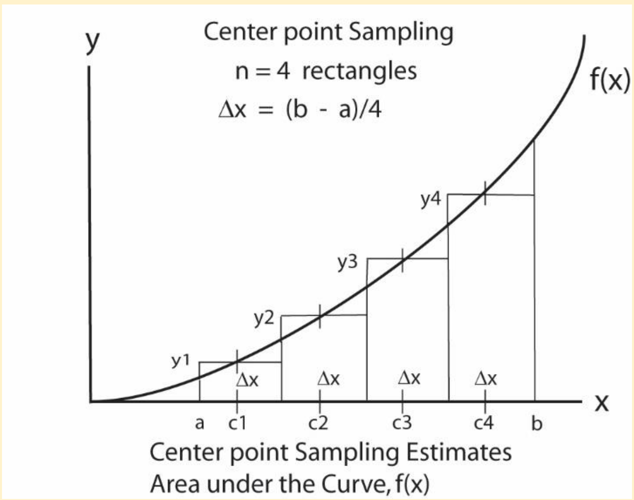
Here $b = 3$, $a = 1$, $n = 4$ so $\Delta x = (3 – 1)/4 = 0.5$

$x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$

$y_0 = 1$, $y_1 = 2.25$, $y_2 = 4$, and $y_3 = 6.25$

Approximate area = (1.0 + 2.25 + 4 + 6.5) 0.5 = 6.75
Exact value of area = 26/3 = 8.66666

Estimate the area under the curve $y(x) = x^2$ from $x = 1$ to $x = 3$ using the center point also called the mid point.
Note: This function happens to be an "increasing" function. (concave up)



Note: The approximate area = (y1 + y2 + y3 + y4) Δx

Further Note: In this application use of midpoint sampling will provide an improved estimation of the actual area.

Here b = 3, a = 1, n = 4 so Δx = (3 – 1)/4 = 0.5

c1 = 1.25, c2 = 1.75, c3 = 2.25, and c4= 2.75 note here $yi = ci^2$

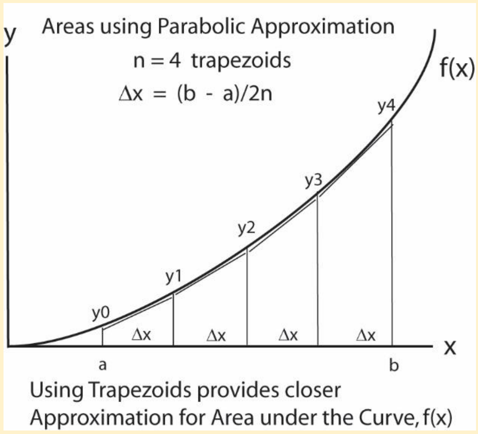
y1 = 1.5625, y2 = 3.0625, y3 = 5.0625, y4 = 7.5625

Approximate area = (1.25 + 1.75 + 2.25 + 2.75) 0.5 = 8.625

Exact value of area = 26/3 = 8.66666

Estimate the area under the curve (x) = x^2 from x = 1 to x = 3 using trapezoids.

Note: This function happens to be an "increasing" function. (concave up)



Here Δx = (b – a)/2n

Note: The approximate area = (y0 + 2y1 + 2y2 + 2y3 + y4) Δx

Here b = 3, a = 1, n = 4 so Δx = (3 – 1)/8 = 0.25

x0 = 1, y1 = 1.25, y2 = 2, y3= 2.5, and x4 = 3

y0 = 1, y1 = 2.25, y2 = 4, y3= 6.25, and y4 = 9

Approximate area = (1.0 + 2x2.25 + 2x4 + 2x6.5 + 9) 0.25 = 8.875

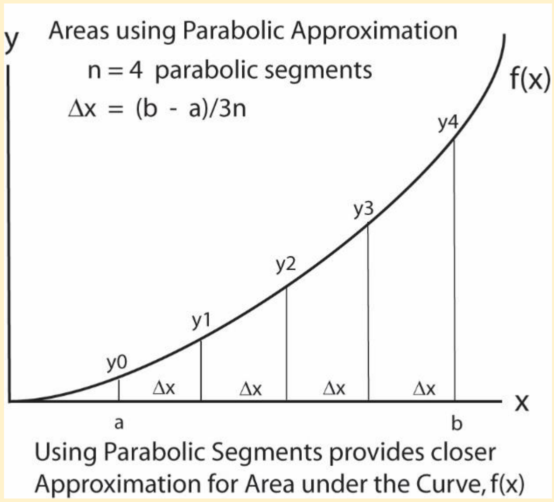
Using trapezoids provides an improved approximation of the actual area.

Exact value of area = 26/3 = 8.66666

Simpson's Rule (Uses parabolas to estimate areas.)

Estimate the area under the curve $y(x) = x^2$ from x = 1 to x = 3 using parabolas.

Note: This function happens to be an "increasing" function. (concave up)



Restriction: You need an even number of segments for the calculation.

Interval of integration is from x = a to x = b

n = number of segments

Here $\Delta x = (b - a)/n$ = width of each segment

Note: The approximate area = (y0 + 4y1 + 2y2 + 4y3 + y4) Δx/3

Also note the pattern of "weights" for each segment.

Here b = 3, a = 1, n = 4 so $\Delta x = (3 - 1)/12 = 0.1666666$

x0 = 1, x1 = 1.25, x2 = 2, x3= 2.5, and x4 = 3

y0 = 1, y1 = 2.25, y2 = 4, y3= 6.25, and y4 = 9

Approximate area = (1.0 + 4x2.25 + 2x4 + 4x6.25 + 9) 0.16666 = 8.66666

Note: Since the function was a parabola, the approximation using parabolic segments yields the exact area, 26/3 = 8.66666.