Limits and Continuity for a function with two independent variables

In a Nut Shell: Suppose f(x,y) approaches the value of L as (x,y) approaches the value of (a,b). Then the limit means that the distance between f(x,y) and L can be made arbitrarily small by making the distance between (x,y) and (a,b) sufficiently small (but not zero).

The formal definition of the limit:

$$\lim_{(x,y) \to (a,b)} f(x,y) = L$$

If for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $(x,y) \in D$ and $0 < \sqrt{[(x-a)^2 + (y-b)^2]} < \delta$ then $| f(x,y) - L < \varepsilon$.

Strategy for finding limits:

It's easiest for cases where the limit does not exist. So first attempt to show that the limit does not exist. In this case, the strategy is to approach the limit of the function from different directions. If any two directions yield a different limit, then the limit of the function does not exist.

If you try several directions and each yields the same limit, then perhaps the limit does exist. This type is much more difficult. Strategies to show the limit exists and find the value of the limit in this case include:

a. If the function is a polynomial or a rational function then use continuity to find its limit.

b. Use the squeeze theorem to determine the limit.

c. Use the formal definition of the limit directly. (as given above)

Squeeze Theorem for functions with two independent variables:

If $f(x,y) \leq g(x,y) \leq h(x,y)$ when (x,y) is near (a,b) except possibly at (a,b) and

$$\lim_{(x,y) \to (a,b)} f(x,y) = \lim_{(x,y) \to (a,b)} h(x,y) = L$$

then $\displaystyle\lim_{(x,y) \to (a,b)} g(x,y) = L$

Example: Find the limit if it exists.

$$\lim_{(x,y)\to(1/2,0)} \ln\left[(1+y^2)/(x^2+xy) \right]$$

Note: $(1+y^2)/(x^2+xy)$ is a rational function and is therefore continuous.

For continuous functions $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

So $\lim_{(x,y)\to(1/2,0)} \left[(1+y^2)/(x^2+xy) \right] = 4$

Also $\ln(s)$ is a continuous function as long as $s > 0$.

Thus $\lim_{(x,y)\to(1/2,0)} \ln\left[(1+y^2)/(x^2+xy) \right] = \ln 4$.

Example: $\lim_{(x,y)\to(0,0)} \left[(x^2 + 2\sin^2 y)/(2x^2+2y^2) \right]$

First evaluate the limit along $y = 0$.

$$\lim_{(x,y)\to(x,0)} \left[x^2 / 2x^2 \right] = 1/2$$

Next evaluate along $x = 0$.

for $y \neq 0$ $\lim_{(0,y)\to(0,y)} \left[2\sin^2 y / 2y^2 \right] = \lim_{(0,y)\to(0,y)} \left[\sin y / y \right]^2$

if $y = 0$ $\lim_{y\to 0} \sin y / y = 1$

Since the values are different for each direction, the limit does not exist.

Find the limit if it exists.

Example: $\lim_{(x,y)\to(0,0)} \left[(x^2 \sin^2 y)/(x^2+2y^2) \right]$

First evaluate the limit along $y = 0$.

$$\lim_{(x,y)\to(x,0)} \left[0 / 2x^2 \right] = 0 \quad \text{for arbitrary } x$$

Next evaluate along $x = 0$.

for $y \neq 0$ $\lim_{(0,y)\to(0,y)} \left[0 / 2y^2 \right] = 0$ for arbitrary y

It appears that the limit might be zero. But we've only checked two directions.
Try using the squeeze theorem to evaluate the limit.

$$0 \le (x^2 \sin^2 y) / (x^2 + 2y^2) \le \sin^2 y$$

since $\sin y \quad 0 \le 1$ and $x^2 / (x^2 + 2y^2) \le 1$

Now $\lim_{y \to 0} \sin^2 y = 0$

So by the squeeze theorem: $0 \le (x^2 \sin^2 y) / (x^2 + 2y^2) \le 0$

And therefore $\lim_{(x,y) \to (0,0)} [(x^2 \sin^2 y) / (x^2 + 2y^2)] = 0$ (result)

Continuity for a function with two independent variables

In a Nut Shell: Continuity means that if the point (x,y) changes by a small amount then the value of f(x,y) also changes by a small amount. It implies that the surface, f(x,y), contains no holes or breaks.

The formal definition of continuity:
A function of two variables, x and y, is continuous at (a,b) if

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$$

f(x,y) is continuous at every point (a,b) in D, its domain.

Types of continuous functions:

- All polynomials in x and y are continuous on the plane.
- The sums, differences, products, and quotients of continuous functions are continuous.
- A rational function is a ratio of polynomials. So any rational function is continuous on its domain, D.

Example: Find the limit if it exists.

$$\lim_{(x,y) \to (1,1)} x^3 y / (2x^4 + 3 y^4)$$

The function $f(x,y) = x^3 y / (2x^4 + 3 y^2)$ is continuous in the plane so

$$f(1,1) = 1/(2 + 3) = 1/5$$

Example: Find the limit if it exists.

$$\lim_{(x,y)\to(0,0)} x^3y / (2x^4 + 3\, y^4)$$

Although $f(x,y) = x^3y / (2x^4 + 3\, y^4)$ is a rational function it is undefined at (0,0).
So substituting in x = 0 and y = 0 does not apply. Instead:

Investigate the limit along y = 0. i.e. the x-axis

For $x \neq 0$ $$\lim_{(0,y)\to(0,y)} x^3y / (2x^4 + 3\, y^2) = 0 / 2x^4 = 0$$

Investigate along y = x

$$\lim_{(0,y)\to(0,y)} x^4 / (2x^4 + 3\, x^4) = 1 / 5$$

Since the limits for these two directions differ, the limit does not exist.

Example: $$\lim_{(x,y)\to(0,0)} [(x^2 \sin^2 y) / (x^2 + 2y^2)]$$

First evaluate the limit along y = 0.

$$\lim_{(x,y)\to(x,0)} [0 / 2x^2] = 0 \quad \text{for arbitrary } x$$

Next evaluate along x = 0.

for $y \neq 0$ $$\lim_{(0,y)\to(0,y)} [0 / 2y^2] = 0 \quad \text{for arbitrary } y$$

It appears that the limit might be zero. But we've only checked to directions.
Try using the squeeze theorem to evaluate the limit.

$$0 \leq (x^2 \sin^2 y) / (x^2 + 2y^2) \leq \sin^2 y$$

since $\sin y \quad 0 \leq 1$ and $x^2 / (x^2 + 2y^2) \leq 1$

Now $$\lim_{y\to 0} \sin^2 y = 0$$

So by the squeeze theorem: $0 \leq (x^2 \sin^2 y) / (x^2 + 2y^2) \leq 0$

And therefore $$\lim_{(x,y)\to(0,0)} [(x^2 \sin^2 y) / (x^2 + 2y^2)] = 0 \quad (\text{result})$$