Fourier Series

In a Nut Shell: A Fourier Series is an infinite series of sine and cosine terms used to represent any periodic function. Suppose f(t) is a periodic function. Then the Fourier Series expansion of f(t) is

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

The objective is to determine the Fourier coefficients, a_o, a_n and b_n .

The Fourier Series representations of the function, f(t), may contain only cosine terms, only sine terms, or both cosine terms and sine terms depending on whether the function, f(t), is an even function, an odd function, or neither even nor odd. More information on details on these terms follows.

Why discuss Fourier Series?

One reason is that it provides a way to represent more complicated (more realistic) forcing functions as, for example, with applications to vibration problems with a forcing function, f(t). Let x = x(t) where x(t) is the displacement of the mass, m, with time t, c is the damping constant, and k is the spring rate. The differential equation of motion for forced vibrations of mass, m, is:

$$m\, d^2x/dt^2 + c\, dx/dt + k\, x = f(t)$$

What is a periodic function? If P is the period of the function, f(t), then the value of f(t) repeats for every period. In other words, f(t) = f (t+ P). The period of a function, f(t), can take on any value.

Case 1: f(t) has a period, P, of 2π. The **Fourier series expansion** of f(t) is then

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

where a_o, a_n, and b_n are the Fourier coefficients.

$$a_o = (1/\pi)\int_{-\pi}^{\pi} f(t)\, dt\,, \quad a_n = (1/\pi)\int_{-\pi}^{\pi} f(t) \cos nt\, dt\,, \quad b_n = (1/\pi)\int_{-\pi}^{\pi} f(t) \sin nt\, dt$$

In a Nut Shell: The Fourier Series expansion for the function, f(t), of period 2L is:

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos n\pi t/L + b_n \sin n\pi t/L$$

where the Fourier coefficients a_o, a_n, and b_n are determined by

$$a_o = (1/L)\int_{-L}^{L} f(t)\,dt, \quad a_n = (1/L)\int_{-L}^{L} f(t)\cos n\pi t/L\,dt \quad b_n = (1/L)\int_{-L}^{L} f(t)\sin n\pi t/L$$

Strategy: Determine if the function, f(t), is an even function, an odd function, or a function that is neither even nor odd. Use this information to determine those coefficients that are zero at the outset.

For "even" functions such as t^2, cos t, t sin t you only need to calculate the Fourier coefficients a_o and a_n with all the b_n's being zero. Note t sin t, is the product of an odd function, t , with another odd function, sin t , which produces an "even" function where **f(t) = f(– t)**.

For "odd" functions such as t, sin t, t cos t you only need to calculate the Fourier coefficients b_n's with a_o and all the a_n being zero. Note t cos t is the product of the odd function, t , with an even function, cos t , which produces an "odd" function where **f(-t) = - f(t)** .

If the function, f(t), is neither even nor odd, then all the Fourier Coefficients need to be calculated.

In summary for f(t) and g(t):

f odd → $a_n = 0$ for all $n \geq 0$ f even and g even → fg even

f even → $b_n = 0$ for all $n \geq 1$ f odd and g odd → fg even

f odd and g even → fg odd or f even and g odd → fg odd

Note for an "even" function, f(t), the product of f(t) cos nπt/L is also "even" . Thus

$$a_o = (1/L)\int_{-L}^{L} f(t)\,dt, \quad a_n = (1/L)\int_{-L}^{L} f(t)\cos n\pi t/L\,dt, \quad b_n = 0$$

$$a_o = (2/L)\int_{0}^{L} f(t)\,dt, \quad a_n = (2/L)\int_{0}^{L} f(t)\cos n\pi t/L\,dt, \quad b_n = 0$$

In summary: f odd $\rightarrow$ $a_n = 0$ for all $n \geq 0$

f even $\rightarrow$ $b_n = 0$ for all $n \geq 1$

Note for an "odd" function, f(t) and f(t) = sin nπt/L which is also "odd"
The product is therefore an even function.

$$b_n = (1/L)\int_{-L}^{L} f(t) \sin n\pi t/L \, dt, \qquad a_0 = a_n = 0$$

$$b_n = (2/L)\int_{0}^{L} f(t) \sin n\pi t/L \, dt, \qquad a_0 = a_n = 0$$

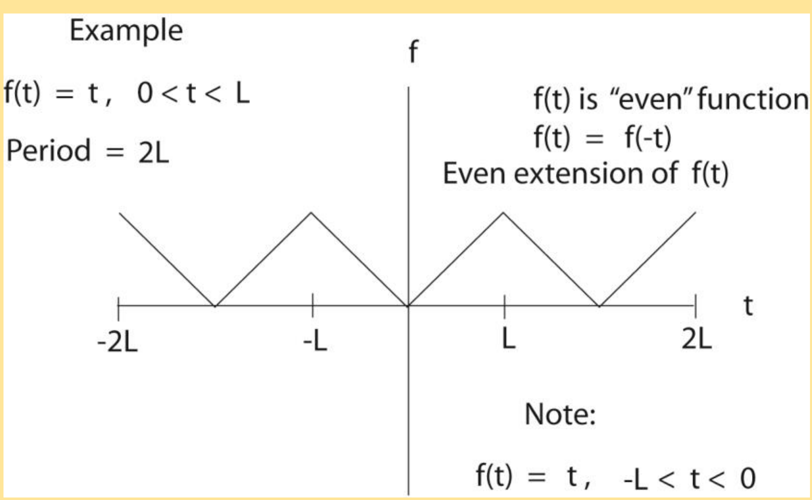
Note for an "even" function, f(t) and f(t) cos nπt/L which is also "even"
The product is therefore an even function.

$$a_o = (1/L)\int_{-L}^{L} f(t)\, dt, \quad a_n = (1/L)\int_{-L}^{L} f(t) \cos n\pi t/L \, dt, \qquad b_n = 0$$

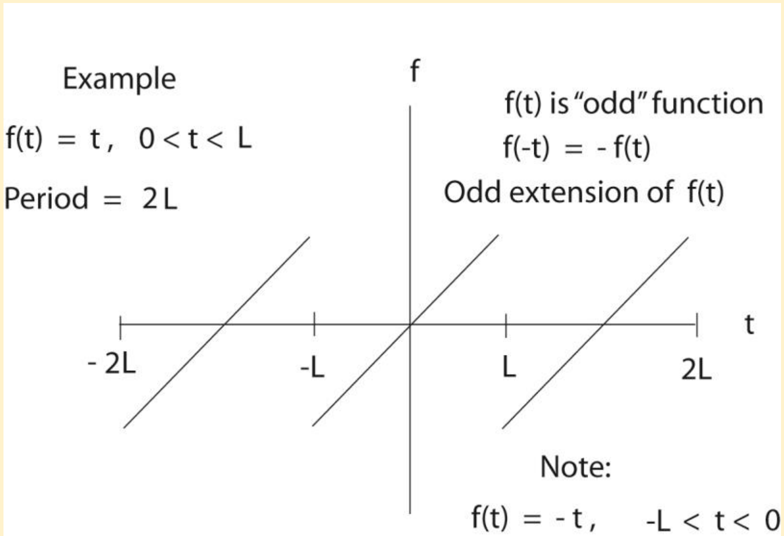
$$a_o = (2/L)\int_{0}^{L} f(t)\, dt, \quad a_n = (2/L)\int_{0}^{L} f(t) \cos n\pi t/L \, dt, \qquad b_n = 0$$

In some cases f(t) is defined only on 0 < t < L and we want to represent f(t) by a Fourier Series of period 2L. **The extension of f(t)** may be represented either as an even or as an odd function.

The figure below illustrates the "even" extension of f(t).



The figure below illustrates the "odd" extension of f(t).



A function, f(t), that is neither odd nor even contains all the terms of the Fourier series.

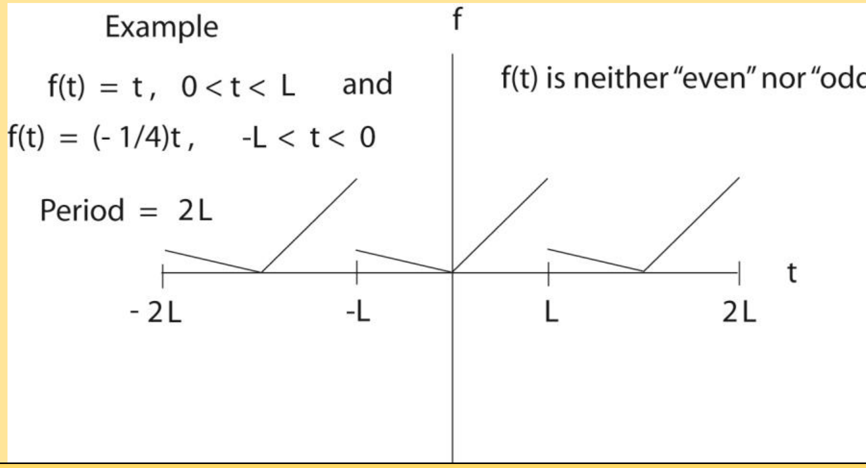
i.e.

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos n\pi t/L + b_n \sin n\pi t/L$$

where the Fourier coefficients a_o , a_n , and b_n are determined by

$$a_o = (1/L) \int_{-L}^{L} f(t)\, dt, \quad a_n = (1/L) \int_{-L}^{L} f(t) \cos n\pi t/L\, dt, \quad b_n = (1/L) \int_{-L}^{L} f(t) \sin n\pi t/L\, dt$$

The figure below shows an example where f(t) has a period of 2L and is neither an odd nor even function.



Strategy: Drop a vertical line at $t = L/2$ and a line at $t = -L/2$ on the plot above.

Note the values of $f(t) \neq f(-t)$ so f(t) is not an "even" function.

Also note that $f(t) \neq -f(-t)$ so f(t) is not an "odd" function.

In this case, f(t), is neither even nor odd.

Suppose f(t) is piecewise continuous on [0,L] of period 2L, then the **Fourier cosine series** of f(t) is

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos n\pi t/L$$

where

$$a_o = (2/L)\int_0^L f(t)\,dt, \qquad a_n = (2/L)\int_0^L f(t)\cos n\pi t/L\,dt$$

Suppose f(t) is piecewise continuous on [0,L] of period 2L, then the **Fourier sine series** of f(t) is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t/L$$

where

$$b_n = (2/L)\int_0^L f(t)\sin n\pi t/L\,dt$$

For an "even" function, f(t), defined by a single formula for $0 < t < 2L$

Period = 2L

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos n\pi t/L$$

where

$$a_o = (2/L)\int_0^L f(t)\,dt, \quad a_n = (2/L)\int_0^L f(t)\cos n\pi t/L\,dt$$

For an "odd" function, f(t), defined by a single formula for $0 < t < 2L$

Period = 2L

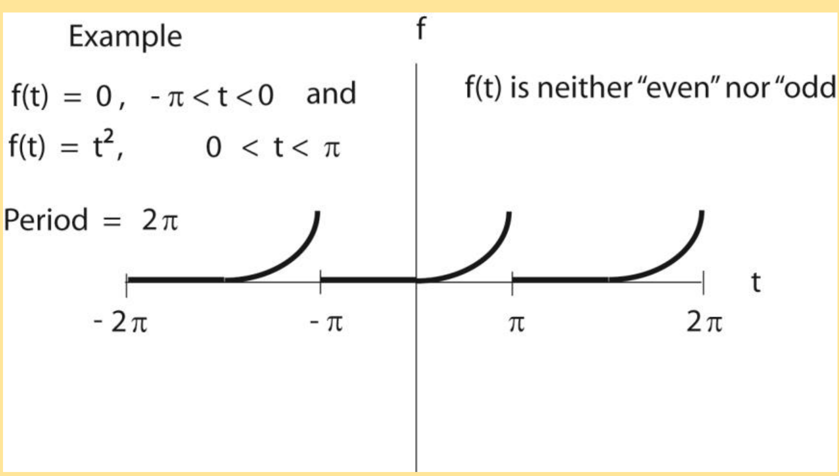
$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\pi t/L$$

where

$$b_n = (1/L)\int_0^{2L} f(t)\sin n\pi t/L\,dt$$

Example: Find the Fourier Series representation of f(t) where

$$f(t) = \begin{cases} 0 & -\pi \le t \le 0 \\ t^2 & 0 \le t \le \pi \end{cases}.$$



The Fourier series expansion of f(t) is:

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

Now calculate the Fourier coefficients a_o, a_n, b_n

Note: The period of f(t) is 2π and f(t) is zero from $-\pi \le t \le 0$. So one needs to only evaluate the integrals from $0 \le t \le \pi$.

$$a_o = (1/\pi) \int_0^{\pi} f(t)\, dt = (1/\pi) \int_0^{\pi} t^2\, dt = (1/\pi)\ t^3/3 \Big|_0^{\pi} = \pi^2/3$$

$$a_n = (1/\pi) \int_0^{\pi} f(t) \cos nt\, dt = (1/\pi) \int_0^{\pi} t^2 \cos nt\, dt$$ now integrate by parts:

$u = (1/\pi)\, t^2 \qquad dv = \cos nt\ dt$

$du = (2t/\pi)\ dt \qquad v = (1/n) \sin nt$

$$a_n = (1/\pi)\ [\,(t^2/n) \sin nt\,]_0^{\pi} - (2/n\pi)\ ) \int_0^{\pi} t \sin nt\, dt = (-2/n\pi)\ ) \int_0^{\pi} t \sin nt\, dt$$

$u = 2t/\pi \qquad dv = -\sin nt\ dt$

$du = 2dt/\pi \qquad v = (1/n) \cos nt$

$$a_n = [\,2t/n^2 \cos nt\,]_0^{\pi} - (2/n^2\pi) \int_0^{\pi} \cos nt\, dt = [\,(2t/n^2\pi) \cos nt\,]_0^{\pi}$$

$$a_n = 2 \cos n\pi / n^2$$

$$b_n = (1/\pi) \int_0^{\pi} f(t) \sin nt\, dt = (1/\pi) \int_0^{\pi} t^2 \sin nt\, dt$$ now integrate by parts:

$$u = (1/\pi)\, t^2 \qquad dv = \sin nt\ dt$$

$$du = 2t\, dt \qquad v = (-1/n) \cos nt$$

$$b_n = (-1/\pi)\ \left[(t^2/n) \cos nt \right]_0^{\pi} + (2/n\pi) \int_0^{\pi} t \cos nt\, dt$$ integrate by parts again

$$u = (2/n\pi)\, t \qquad dv = \cos nt\ dt$$

$$du = (2/n\pi)\, dt \qquad v = (1/n) \sin nt$$

$$b_n = (-1/\pi)\ \left[t^2/n \cos nt \right]_0^{\pi} + \left[(2/n^2\pi)\, t \sin nt \right]_0^{\pi} - (2/n^2\pi) \int_0^{\pi} \sin nt\, dt$$

$$b_n = -(\pi/n) \cos n\pi + \left[(2/n^3\pi) \cos nt \right]_0^{\pi}$$

so

$$b_n = -(\pi/n)\cos n\pi + (2/n^3\pi)\, [\cos n\pi - 1]$$

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

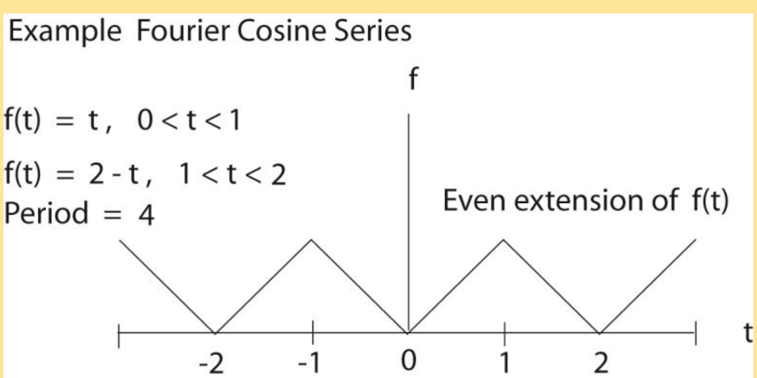
Now substitute in for the Fourier coefficients a_o, a_n, and b_n

$$f(t) = \pi^2/6 + \sum_{n=1}^{\infty} (2 \cos n\pi / n^2) \cos nt + [2(\cos n\pi - 1)/n^3\pi - (\pi \cos n\pi)/n] \sin nt$$

which is the Fourier series expansion for the given f(t)

Example: Find the Fourier "cosine series" expansion for f(t) where

$$f(t) = \begin{cases} t & 0 < t < 1 \\ 2 - t & 1 < t < 2 \end{cases}$$ with period 4 of even extension



Note: The Fourier Cosine series of period 4 even expansion of f(t) contains only the terms:

a_o and the a_n 's. All the b_n 's are zero.

The Fourier cosine series expansion of f(t) is:

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi t/2)$$

Recall that if f(t) is piecewise continuous on [0,L], then the **Fourier cosine series** of f(t) is

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos n\pi t/L$$

where

$$a_o = (2/L) \int_0^L f(t)\,dt, \qquad a_n = (2/L) \int_0^L f(t) \cos n\pi t/L\,dt$$

So in this example: P = 4 and L = 2

and the Fourier coefficients become

$$a_o = (2/2) \int_0^2 f(t)\,dt = \int_0^1 t\,dt + \int_1^2 (2 - t)dt \;| = t^2/2\,\Big|_0^1 + (2t - t^2/2)\Big|_1^2 = 1$$

Next calculate the a_n 's

$$a_n = (2/2) \int_0^1 t \cos(n\pi t/2)\,dt + 2/2 \int_1^2 (2 - t)\cos(n\pi t/2)\,dt \;\Big|_1^2 = I_1 + I_2 + I_3$$

Use integration by parts to evaluate the integrals I_1 and I_3

$$a_n = (2/2) \int_0^1 t \cos(n\pi t/2)\,dt + 2/2 \int_1^2 (2 - t)\cos(n\pi t/2)\,dt \;\Big|_1^2 = I_1 + I_2 + I_3$$

Use integration by parts to evaluate the integrals I_1 and I_3

For I_1

$$u = t \qquad dv = \cos(n\pi t/2)\,dt$$

$$du = dt \qquad v = (2/n\pi)\sin(n\pi t/2)$$

$$a_{n1} = [(2t/n\pi)\sin(n\pi t/2)]_0^1 - (2/n\pi)\ )\int_0^1 \sin(n\pi t/2)\,dt =$$

$$a_{n1} = (2t/n\pi)\sin(n\pi t/2)\Big|_0^1 + (4/n^2\pi^2)\cos(n\pi t/2)\Big|_0^1 =$$

$$a_{n1} = 2/n\pi)\sin(n\pi/2) - (4/n^2\pi^2)[\cos(n\pi/2) - 1]$$

For I_2 $a_{n2} = \int_1^2 2\cos(n\pi t/2)\,dt = (4/n\pi)\sin(n\pi t/2)\Big|_1^2 = (-4/n\pi)\sin(n\pi/2)$

For I_3 the calculation of a_{n3} is similar to a_{n1} using a minus sign and change of limits from 0 to 1 to 1 to 2. The result is:

$$a_{n3} = -(2t/n\pi)\sin(n\pi t/2)\Big|_1^2 - (4/n^2\pi^2)\cos(n\pi t/2)\Big|_1^2 =$$

$$a_{n3} = -(4/n\pi)\sin(n\pi) + (2/n\pi)\sin(n\pi/2) - (4/n^2\pi^2)[\cos n\pi - \cos(n\pi/2)]$$

Now collect terms:

$a_n = a_{n\,1} + a_{n2} + a_{n3} =$

$(2/n\pi)\sin(n\pi/2) - (4/n^2\pi^2)[\cos(n\pi/2) - 1] + (-4/n\pi)\sin(n\pi/2) +$

$-(4/n\pi)\sin(n\pi) + (2/n\pi)\sin(n\pi/2) - (4/n^2\pi^2)[\cos n\pi - \cos(n\pi/2)]$

Strategy: Group like terms.

$(2/n\pi)\sin(n\pi/2) - (4/n\pi)\sin(n\pi/2) + (2/n\pi)\sin(n\pi/2)$ (these terms cancel)

Leaving:

$$a_n = (4/n^2\pi^2)[\cos(n\pi/2) - 1] - (4/n^2\pi^2)[\cos n\pi - \cos(n\pi/2)]$$

or $$a_n = (8/n^2\pi^2)\cos(n\pi/2) - 4/n^2\pi^2 - (4/n^2\pi^2)\cos n\pi$$

$$a_n = (8/n^2\pi^2)\cos(n\pi/2) - 4/n^2\pi^2\,(1 + \cos n\pi)$$

$$a_n = (8/n^2\pi^2)\cos(n\pi/2) - 4/n^2\pi^2\,(1 + (-1)^n)$$

So for n odd $a_n = 0$ and for n even $a_n = - 8/n^2\pi^2) [1- \cos(n\pi/2)]$

But $1 - \cos(n\pi/2) = 2 \sin^2(n\pi/4)$ So $a_n = - (16/n^2\pi^2) \sin^2(n\pi/4)$

Thus, the Fourier cosine series expansion of f(t) is:

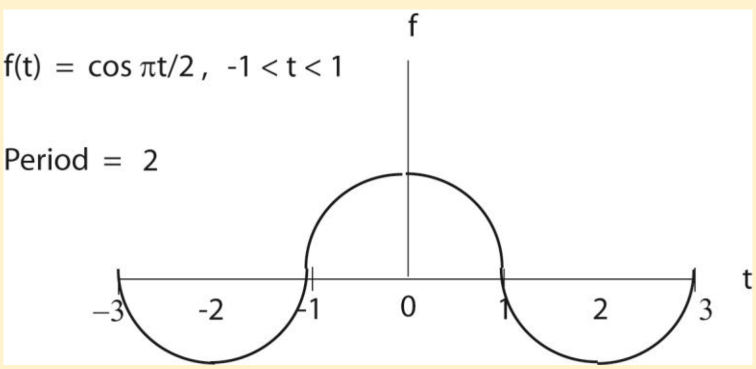
$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi t/2)$$

$$f(t) = ½ - 16/\pi^2 \sum_{n \text{ even}} (1/n^2)\sin^2(n\pi/4) \cos(n\pi t/2)$$

Example: Find the Fourier series representation of f(t) using a complex notation approach where

$f(t) = \cos \pi t/2 \quad -1 < t < 1$ Shown below.

Recall that the cosine function is an "even" function i.e. $f(t) = f(-t)$



Here P = period = 2 = 2L , so L = 1 and 1/L = 1

L represents half of one period

Since f(t) is an even function, the only nonzero Fourier coefficients should be the

a_o and a_n's All the b_n's should be zero.

$$a_n = \int_{-1}^{1} \cos \pi t/2 \, dt = (2/\pi) \sin \pi t/2 \Big|_{-1}^{1} = (2/\pi) [\sin \pi/2 - \sin (-\pi/2)] = 4/\pi$$

Note that $\exp(in\pi t) = \cos n\pi t + i \sin n\pi t$

$$a_n + i b_n = \int_{-1}^{1} f(t) \exp(in\pi t) \, dt = \int_{-1}^{1} \cos \pi t/2 \, \exp(in\pi t) \, dt$$

$$u = \cos \pi t/2 \qquad dv = \exp(in\pi t) \, dt$$

$$du = (-\pi/2) \sin \pi t/2 \, dt \qquad v = (1/in\pi) \exp(in\pi t)$$

$$a_n + i b_n = \cos \pi t/2 \, \exp(in\pi t) / in\pi \Big|_{-1}^{1} + (1/2in) \int_{-1}^{1} \sin \pi t/2 \, \exp(in\pi t) \, dt$$

$$u = (1/2in) \sin \pi t/2 \qquad dv = \exp(i\pi t) \, dt$$

$$du = (\pi/4in) \cos \pi t/2 \, dt \qquad v = (1/in\pi) \exp(i\pi t)$$

$$a_n + i b_n = \cos \pi t/2 \, \exp(i\pi t) / in\pi \Big|_{-1}^{1} + (1/2in)(1/in\pi) \sin \pi t/2 \exp(in\pi t) \Big|_{-1}^{1}$$

$$- (\pi/4in)(1/in \, \pi) \int_{-1}^{1} \cos \pi t/2 \exp(in\pi t) \, dt$$

$$a_n + i b_n = (-1/2n^2 \pi) \sin \pi t/2 \, \exp(i\pi t) \Big|_{-1}^{1} + (1/4n^2) \int_{-1}^{1} \cos \pi t/2 \exp(in\pi t) \, dt$$

But

$$a_n + i b_n = \int_{-1}^{1} \cos \pi t/2 \exp(in\pi t) \, dt$$

So collecting terms yields (bring the integral on the rhs to the lhs)

$$[\,1 - 1/4n^2\,]\,[a_n + i\,b_n\,] = (-1/2n^2\,\pi)[\sin(\pi/2)\,\exp(in\pi) - \sin(-\pi/2)\,\exp(-in\pi)\,]$$

$$[\,1 - 1/4n^2\,]\,[a_n + i\,b_n\,] = (-1/2n^2\,\pi)[\sin(\pi/2)\,\exp(in\pi) - \sin(-\pi/2)\,\exp(-in\pi)\,]$$

$$[\,1 - 1/4n^2\,]\,[a_n + i\,b_n\,] = (-1/2n^2\,\pi)[\sin(\pi/2)\,\exp(in\pi) - \sin(-\pi/2)\,\exp(-in\pi)\,]$$

$$[\,1 - 1/4n^2\,]\,[a_n + i\,b_n\,] = (-1/2n^2\,\pi)\,[\exp(in\pi) + \exp(-in\pi)]$$

$$[(4n^2 - 1)/4n^2\,]\,[a_n + i\,b_n\,] = (-1/2n^2\,\pi)\,[\cos n\,\pi + i\sin n\,\pi + \cos n\,\pi - i\sin n\,\pi\,]$$

Collect terms and solve for $a_n + i\,b_n$,

$$a_n + i\,b_n = [4n^2/4n^2 - 1)]\,[-\,1/2n^2\,\pi\,]\,[\,2\cos n\,\pi\,]$$

$$a_n + i\,b_n = -\,(4/\pi)\cos n\,\pi\,/\,(4n^2 - 1)$$

Next equate the real and imaginary parts to obtain:

$$a_n = -\,(4/\pi)\cos n\,\pi\,/\,(4n^2 - 1) \quad \text{and} \quad b_n = 0$$

Now

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

So

$$f(t) = 2/\pi - \sum_{n=1}^{\infty} [(4/\pi)\cos n\,\pi\,/\,(4n^2 - 1)]\cos nt$$

which is the Fourier series expansion for the given f(t)

Now

$$f(t) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt$$

So

$$f(t) = 2/\pi - \sum_{n=1}^{\infty} [(4/\pi)\cos n\,\pi\,/\,(4n^2 - 1)]\cos nt$$

which is the Fourier series expansion for the given f(t)

Note: The complex representation approach yields both a_n and b_n .