First Order Linear Differential Equations, Integrating Factor

In a Nut Shell: There are two common methods to solve first order, linear, ordinary differential equations. They include:

- **Separation of variables** (easiest if the dependent and independent variables separate.)
- **Find an integrating factor**.

Review:

A differential equation, **y ' + p(x)y = q(x)** is said to be linear if its dependent variable y and its derivative y ' are linear.

(Linear = No nonlinear terms involving y or y ')

Example of a first order linear, ordinary d.e. - separable form

dy/dx = y' = f(x) This is the differential equation.

y(a) = A = constant = This is the initial condition on y the dependent variable

Solve by direct integration of dy = f(x) dx (separated form)

$y(x) = \int f(x)\, dx + C$ where C is determined using the initial condition

Example: Solve the d.e. $dy/dx + y \sec^2 x = 0$

Strategy: Use separation of variables. i.e. separate y with x.

Rewrite the d.e. as follows: $dy/dx = - y \sec^2 x$

Next separate variables as follows: $dy/y = - \sec^2 x \, dx$

Integrate both sides of the d.e. :

$\ln y = - \tan x + C_1$ where C_1 is the constant of integration

Solve for y(x). $y(x) = C e^{- \tan x}$

Suppose the initial condition is y(0) = 1, then

In this case 1 = C giving $y(x) = e^{-\tan x}$ (result)

Check:

One nice thing about d.e.'s is you can check your answer to verify that your solution, y(x), satisfies both the d.e. and the initial condition, in this case, $y(0) = 1$, by taking the derivative of y(x)

$dy/dx = - \sec^2 x \; e^{-\tan x} = - \sec^2 x \; y(x)$

or $dy/dx + \sec^2 x \; y(x) = 0$ which is the original d.e.

and $y(0) = 1$ (Check)

Next look at the method of solution using an **Integrating factor**.

NOTE: Every first order, linear, ordinary d.e. has an integrating factor.

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Integrating factor (IF) is as follows: $IF = e^{\int p\backslash(x)\, dx}$

Constant of integration does not matter here.

$e^{\int p(x)\, dx} \; y' + p(x)\, e^{\int p(x)\, dx} \; y = q(x)\, e^{\int p(x)\, dx}$

$d/dx\, [e^{\int p(x)\, dx} \; y] = q(x)\, e^{\int p(x)\, dx}$

$e^{\int p(x)\, dx} \; y = \int [q(x)\, e^{\int p(x)\, dx}]\, dx$

$y(x) = e^{-\int p(x)\, dx} \{ q(x)\, e^{\int p(x)\, dx} \}dx + C$ (solution)

Here the constant of integration matters.

Example: Solve the d.e. $\mathbf{y' + y = e^{2x}}$

with the initial condition: $y(0) = 1$

Strategy: Use an integrating factor (IF).

The general form of a linear, first order, d.e. is $\mathbf{y' + p(x)\, y = q(x)}$

In this example: $p(x) = 1$,

So the integrating factor is $\mathbf{IF = e^{\int dx} = e^{x}}$

Next Step: Multiply each term of the original d.e. by the integrating factor. The result is

$$e^x y' + e^x y = e^{3x}$$

Next Step: Factor terms on the left hand side as follows:

$$d/dx [e^x y] = e^{3x}$$

NOTE: If you calculated the correct IF, then you should have arrived with an expression on the left hand side of the d.e. that you can integrate.

(This result serves as a check on your IF.)

$$e^x y = \int e^{3x} dx = (1/3) e^{3x} + C$$

$y(x) = (1/3) e^{2x} + C e^{-x}$ where C is a constant to be determined from a specified initial condition such as

$y(0) = 1$

In this case $1 = 1/3 + C$, So $C = 2/3$

And $y(x) = (1/3) e^{2x} + (2/3) e^{-x}$ (result)