Surface Integrals

In a Nut Shell: Two questions are relevant. **What** is a surface integral and **how** do you evaluate a surface integral?

Recall the line integral, I, provides the value of a function, f(x,y,z), evaluated along a curve, C, in space. Here the line integral is $I = \int_C f(x, y, z)\, ds$

where ds is the arc length along the curve, C

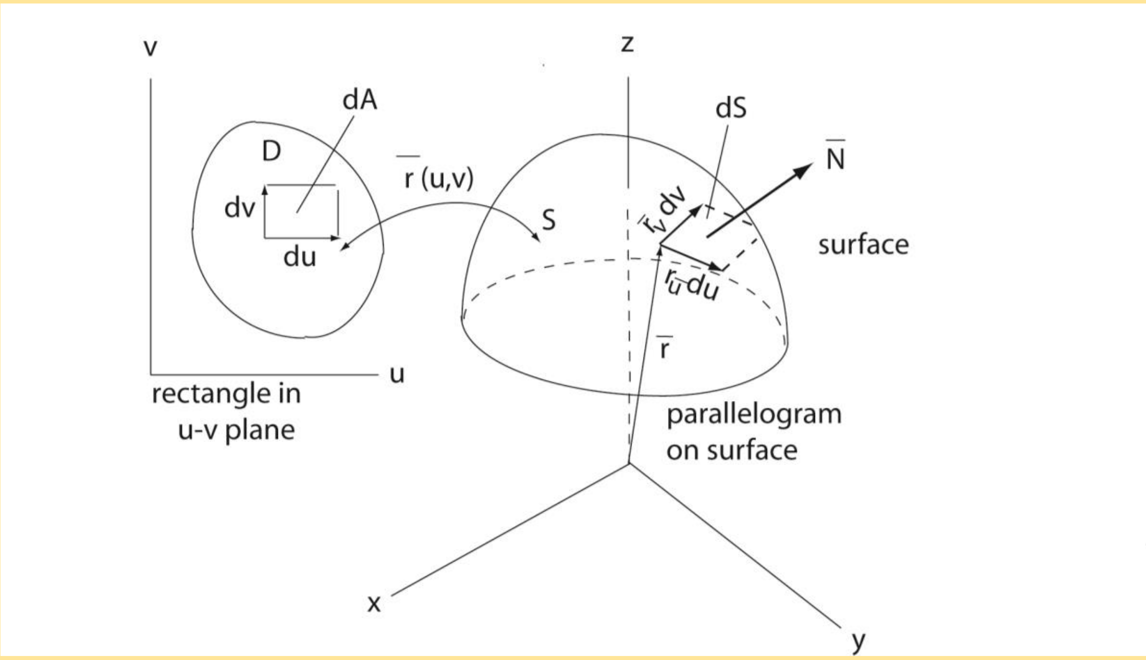
The surface integral, I_s, is analogous to the line integral in that it provides the value of a function, f(x,y,z), evaluated over a "smooth" surface, S, in space. Here the surface integral is $I_s = \int\int f(x, y, z)\, dS$ where dS is the element of surface area on the spatial surface

How do you evaluate a surface integral?

Usually the surface, S, in space is somewhat complicated. So **one strategy** is to transform the element of surface area, dS, from the x-y-z space into a parallelogram dA = du dv in the u-v plane as shown in the figure below.

where **r**(u,v) is the parametric representation of the surface, S.

r(u,v) = < x(u,v), y(u,v), z(u,v) > is the position vector to point on surface, S



Also note the partial derivatives $\mathbf{r}_u = \partial \mathbf{r} / \partial u$ and $\mathbf{r}_v = \partial \mathbf{r} / \partial v$

Now the element of surface area is $dS = | \mathbf{r}_u du \times \mathbf{r}_v dv | = | \mathbf{r}_u \times \mathbf{r}_v | du\, dv$

but $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{N}$ = normal to the surface, S and $du\, dv = dA$

So the relation between the element of surface area, dS, and the element of area, dA is

$dS = | \mathbf{N} | dA$ (See the figure above)

Evaluation of the surface integral $I_s = \int_S f(x, y, z)\, dS$ is a two-step process.

- First write the integrand f(x,y,z) as a function of two independent variables.

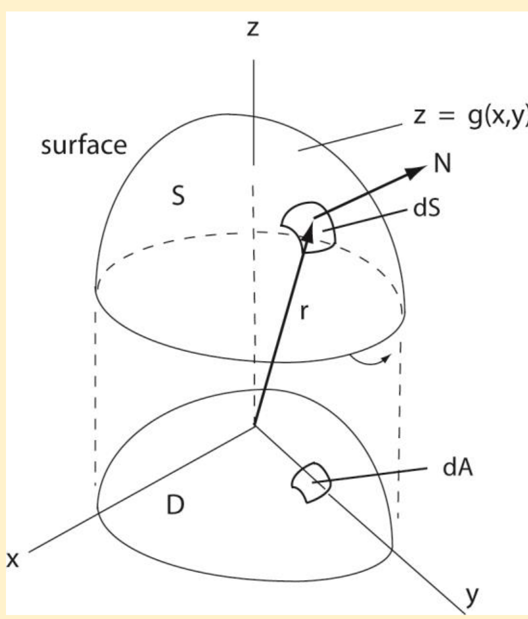
  i.e. If the surface is given by $z = g(x,y)$, then $f(x,y,z) = f(x, y, g(x,y))$

- Next write the element of surface area, dS, in terms of the area element, dA.

$dS = | \mathbf{N}(u,v)|\, du\, dv = | \mathbf{N}(u,v)|\, dA$ where $\mathbf{N}(u,v) = \partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v$

take $u = x$ and $v = y$ so $\mathbf{N}(x,y) = \partial\mathbf{r}/\partial x \times \partial\mathbf{r}/\partial y$

Here $\mathbf{N}(x,y)$ represents the normal vector to the surface, S, at each arbitrary point (x,y,z) and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + g(x,y)\mathbf{k}$ is the position vector from the origin of the coordinate system to any arbitrary point (x,y,z) on the surface, S. See the figure below.



Note: Domain D shown in the previous figure is the "area" projected on to the relevant plane. In this case it is projected on to the x-y plane since the surface, S, was described by $z = g(x,y)$.

When the surface, S, is described by $z = g(x,y)$ (recall let $u = x$ and $v = y$) the position vector to an arbitrary point on S is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + g(x,y)\mathbf{k}$

Then one can use x and y rather than u and v as the parameters describing the surface, S. So the normal vector to S is $\mathbf{N}(u,v) = \mathbf{N}(x,y) = \mathbf{r}_x \times \mathbf{r}_y$ which yields the Following 3 x 3 determinant. **Note: N is not a unit vector.**

$$\mathbf{N}(x,y) = \det \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x/\partial x & \partial y/\partial x & \partial g/\partial x \\ \partial x/\partial y & \partial y/\partial y & \partial g/\partial y \end{matrix} = \det \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \partial g/\partial x \\ 0 & 1 & \partial g/\partial y \end{matrix}$$

$$\mathbf{N}(x,y) = -\partial g/\partial x\,\mathbf{i} - \partial g/\partial y\,\mathbf{j} + 1\,\mathbf{k} \quad \text{and} \quad |\mathbf{N}(x,y)| = \sqrt{[1 + (\partial g/\partial x)^2 + (\partial g/\partial x)^2]}$$

Thus $dS = \sqrt{[1 + (\partial g/\partial x)^2 + (\partial g/\partial x)^2]}\, dx\, dy$ and

$$I_S = \iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x,y)) \sqrt{[1 + (\partial g/\partial x)^2 + (\partial g/\partial x)^2]}\, dx\, dy$$

If, on the other hand, the surface was described by $y = g(x,z)$, then D would be the "area" projected on to the x-z plane, etc.

Options: You have two options in evaluating surface integrals.

Option 1: You could attempt direct evaluation of the surface integral

$$I_S = \int\int_S f(x,y,z)\, dS$$

Option 2: You could use a transformation $dS = |\, \mathbf{r}_u \times \mathbf{r}_v \,|\, dA$

$$I_S = \int\int_S f(x,y,z)\, dS = \int\int_D f(x,y,z)\, |\, \mathbf{r}_u \times \mathbf{r}_v \,|\, dA$$

Strategy for Surface Integrals

In a Nut Shell: The strategy used to evaluate surface integrals depends on whether you use direct evaluation of the integral or you choose to use the method of transformation.

Note: If the surface, S, is described by $z = g(x,y)$ then you will integrate in domain D in the x-y plane.

On the other hand, if the surface is described by $y = g(x,z)$, then D would be the "area" projected on to the x-z plane and integration will be in the x-z plane. etc.

Strategy for Option 1: Use direct evaluation of the surface integral.

$$I_S = \iint_S f(x,y,z)\ dS = \iint_S \mathbf{F} \cdot \mathbf{n}\ dS = \iiint_E \text{div}(\mathbf{F})\ dV \quad \text{(from Divergence Theorem)}$$

where $f = f(x,y,z)$ = function of x,y,z whose domain includes the surface, S
$\mathbf{F} = \mathbf{F}(x,y,z)$ is a vector field
$\mathbf{n}$ = unit vector normal to the surface, S
dS = element of area on the surface, S

- Construct a view of surface, S, showing element of surface area, dS.
- Express dS in terms of parameters on the surface of S.
- Express f(x,y,z) in terms of parameters on the surface, S.
- Evaluate $\iint_S f(x,y,z)\ dS$

Strategy for Option 2: You could use a transformation $dS = |\mathbf{r}_u \times \mathbf{r}_v|\ dA$

$$I_S = \iint_S f(x,y,z)\ dS = \iint_D f(x,y,z)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA$$

- Construct a view of surface, S.
- Show the position vector from the origin to an arbitrary point on surface, S.

$$\mathbf{r} = \langle x, y, z \rangle$$

- Determine which plane to project S on giving domain D. i.e.
  If the surface, S, is given by $z = g(x,y)$ then express the position vector as:

  $$\mathbf{r} = \langle x, y, g(x,y) \rangle$$

  and the projection will be on the x-y plane to determine domain, D.

  Note: If the surface, S, was given as $x = g(y,z)$ then

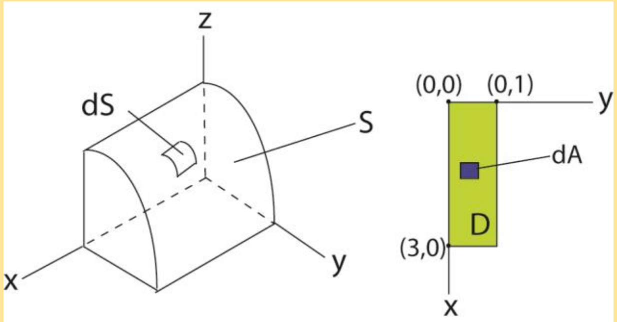
  $$\mathbf{r} = \langle g(y,z), y, z) \rangle \text{ and the projection is on the y-z plane}$$

- In this case for $z = g(x,y)$, calculate $\mathbf{r}_x$ and $\mathbf{r}_y$, $\mathbf{r}_x \times \mathbf{r}_y$, and $|\mathbf{r}_x \times \mathbf{r}_y|$
- Use: $dS = |\mathbf{r}_x \times \mathbf{r}_y|\ dA$
- Evaluate: $\iint_D f(x,y,z)\ |\mathbf{r}_u \times \mathbf{r}_v|\ dA$

Note: $|\mathbf{r}_u \times \mathbf{r}_v|$ is not a unit vector

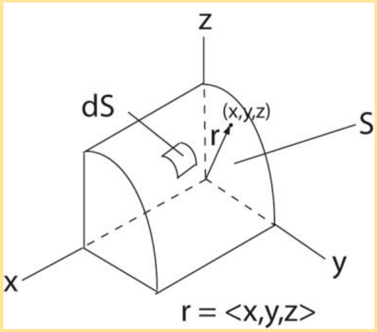
Example: Evaluate the surface integral $I_S = \int\int (z + x^2 y) \, dS$

where S is the part of the cylinder $y^2 + z^2 = 1$ that lies in the first quadrant and bounded by the planes $x = 0$ and $x = 3$. See the figure below.



Strategy: Use a transformation to evaluate the surface integral.

Step 1: Show and express the position vector, **r**, from the origin to and arbitrary point on the surface, S. Projection of S is on to the x- y plane as shown above in the figure on the right.



$$\mathbf{r} = < x, y, z > = < x, y, \sqrt{(1 - y^2)} >$$

Step 2: Calculate the normal to the surface, S. $\mathbf{N} = \mathbf{r_x} \times \mathbf{r_y}$.

$\mathbf{r_x} = < 1, 0, 0 >$ $\mathbf{r_y} = < 0, 1, - y / \sqrt{(1 - y^2)} >$

Step 3: Calculate the normal to the surface, S. $\mathbf{N} = \mathbf{r_x} \times \mathbf{r_y}$.

$$\mathbf{r_x} \times \mathbf{r_y} = < 0, y / \sqrt{(1 - y^2)} , 1 >$$

So $| \mathbf{N} | = \sqrt{1 / (1 - y^2)}$

Step 4: Write the surface area element, dS in terms of dA. $dS = |\mathbf{N}|\, dA$

$$dS = \sqrt{1/(1 - y^2)}\ dx\, dy$$

So

$$I = \iint_S (z + x^2 y)\ dS = \int_{y=0}^{y=1} \int_{x=0}^{x=3} (\sqrt{(1 - y^2)} + x^2 y)\ \sqrt{1/(1 - y^2)}\ dx\, dy$$

Step 5: Evaluate the integral over the domain, D.

$$I_D = \int_{y=0}^{y=1} \int_{x=0}^{x=3} [\ (1 + \ x^2 y / \sqrt{1/(1 - y^2)}\ ]\ dx\, dy$$

Step 6: Perform integration.

$$I_D = \int_{y=0}^{y=1} \int_{x=0}^{x=3} [\ dx\, dy + \int_{y=0}^{y=1} \int_{x=0}^{x=3} \sqrt{(1 + x^2 y / \sqrt{1/(1 - y^2)}\ ]\ dx\, dy$$

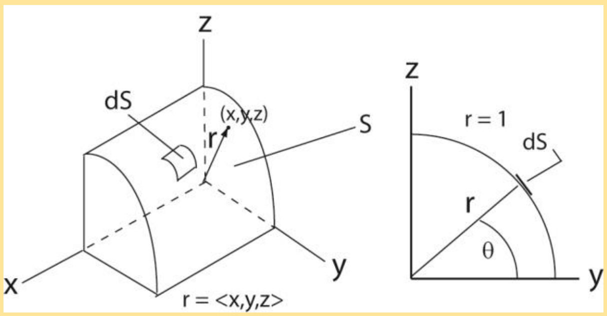
Hint: Use the substitution $w = 1 - y^2$ to help with integration.

$$I_D = 3 + 9 = 12 \quad \text{(result)}$$

Example: Evaluate the surface integral $I_S = \iint (z + x^2 y)\, dS$

where S is the part of the cylinder $y^2 + z^2 = 1$ that lies in the first quadrant

and bounded by the planes $x = 0$ and $x = 3$. See the figure below.



Strategy: Use direct evaluation of the surface integral.

Steps1 and 2: Show surface and express dS in terms of parameters on the surface, S.

$$S: \quad y^2 + z^2 = 1$$

From the figures shown above: $dS = (1)\, d\theta\, dx$

Step 3: Express f(x,y,z) in terms of parameters on S.

From the figure above on the right: $z = \sin\theta$ and $y = \cos\theta$

So $f(x,y,z) = z + x^2 y = \sin\theta + x^2 \cos\theta$

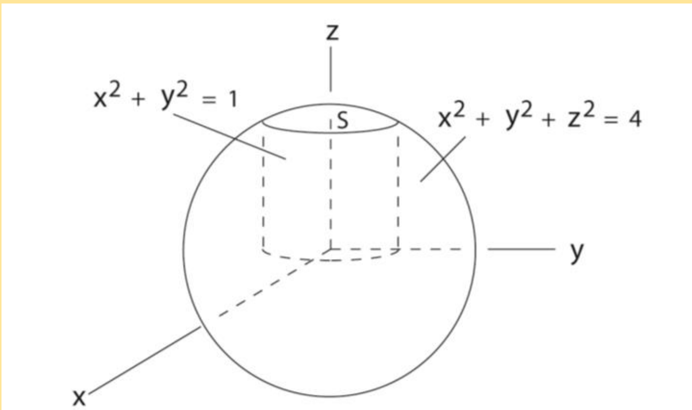
Step 4: Evaluate the surface integral.

$$I = \int_{x=0}^{x=3} \int_{\theta=0}^{\theta=\pi/2} (\sin\theta + x^2 \cos\theta)\, d\theta\, dx$$

The result is $I = 12$ (same result as from using option 2, transformation)

Example: Evaluate the surface integral $I_S = \int\int y^2\, dS$

where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane. See the figure below.

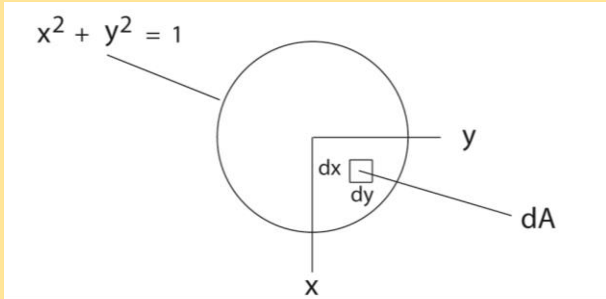


Strategy: Use a two step process.

Step 1: Express f(x,y,z) in terms of the independent variables (in this case) x and y.

So in this example $f(x,y,z) = y^2$

Step 2: Write the surface area element, dS in terms of dA. $dS = |\mathbf{N}|\, dA$
In this example the projection of dS is a circle of radius 1 in the xy-plane.



Next calculate **N :** $\mathbf{N}(x,y) = \mathbf{r}_x \times \mathbf{r}_y$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

In this example $\mathbf{N}(x,y) = \mathbf{r}_x \times \mathbf{r}_y$ where $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

For the surface, S, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x\,\mathbf{i} + y\,\mathbf{j} + [\sqrt{(4 - x^2 - y^2)}]\,\mathbf{k}$

$\mathbf{r}_x = \mathbf{i} + 0\,\mathbf{j} + [-x / \sqrt{(4 - x^2 - y^2)}]\,\mathbf{k}$

$\mathbf{r}_y = 0\,\mathbf{i} + 1\,\mathbf{j} + [-y / \sqrt{(4 - x^2 - y^2)}]\,\mathbf{k}$

Use the cross product to find the normal vector, **N**, to the surface, S:

$$\mathbf{N}(x,y) = \det \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & [-x / \sqrt{(4 - x^2 - y^2)}] \\ 0 & 1 & [-y / \sqrt{(4 - x^2 - y^2)}] \end{matrix}$$

Expansion of this determinant gives

$\mathbf{N}(x,y) = [\, x / \sqrt{(4 - x^2 - y^2)}]\,\mathbf{i} + [\, y / \sqrt{(4 - x^2 - y^2)}]\,\mathbf{j} + \mathbf{k}$

Thus $|\,\mathbf{N}\,| = \sqrt{[\,4 / (4 - x^2 - y^2)]}$

So the surface integral $I_S = \int f(x,y, z(x,y))\, N\, dA$ becomes

$$I_S = \iint_D y^2 \sqrt{[\,4 / (4 - x^2 - y^2)]}\, dA = \iint_D y^2 \sqrt{[\,4 / (4 - x^2 - y^2)]}\, dx\, dy$$

Now $dA = dx\, dy = r\, dr\, d\theta$ (in terms of polar coordinates)
To simplify the integration switch to polar coordinates where $y = r \sin\theta$.

Thus $$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (r \sin\theta)^2 \sqrt{[\,4 / (4 - r^2)\,]}\, r\, dr\, d\theta$$ or

Thus $$\int_{\theta=0}^{\theta=2\pi} [\, \sin^2\theta \int_{r=0}^{r=1} \sqrt{[\,4 / (4 - r^2)\,]}\, r^3\, dr\, d\theta$$ or

To evaluate this integral use $\sin^2\theta = (1 - \cos 2\theta) / 2$ for integration on θ and substitute $r = 2 \sin\varphi$ for the integration on r.

The result is $I_S = \pi\,(\,32/3 - 6\sqrt{3}\,)$

Example: Evaluate the same surface integral $I_S = \int\int y^2 \, dS$
using a **direct approach**

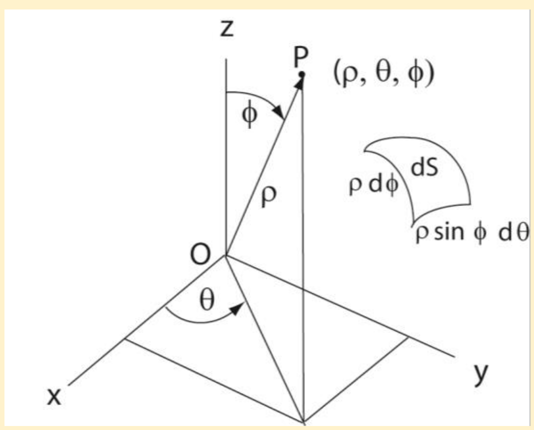
where S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the

cylinder $x^2 + y^2 = 1$ and above the xy-plane. See the figure below.

The surface, S, in this example is just the "cap" of the sphere - - the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.
So **direct evaluation using spherical coordinates may provides an alternate solution.**

For spherical coordinates: $x = \rho \sin\varphi \cos\theta$, $y = \rho \sin\varphi \sin\theta$, and $z = \rho \cos\varphi$

From the figure below we find the expression for the element of surface area, dS,
in spherical coordinates as $dS = \rho \, d\varphi \; \rho \sin\varphi \, d\theta$



In this example $\rho = 2$ so $dS = 4 \sin\varphi \, d\varphi \, d\theta$

Next we must find the limits of integration for the intersecting sphere and cylinder.

From $x^2 + y^2 = 1 = (\rho \sin\varphi \cos\theta)^2 + (\rho \sin\varphi \sin\theta)^2 = \rho^2 \sin^2\varphi$ (cylinder)

Since the radius of the sphere is 2 we get $1 = 4 \sin^2\varphi$. Thus $\sin\varphi = \pm ½$

So the limits of integration on φ are 0 to $\pi / 6$ and the limits of integration on θ are 0 to 2π.
The surface integral, I_S, becomes

Thus $$\int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi/6} (4 \sin^2\varphi \sin^2\theta) \, 4 \sin\varphi \, d\varphi \, d\theta$$

This integration yields the same result as before. $I_S = \pi\left(32/3 - 6\sqrt{3}\right)$

Surface Integrals with "oriented surfaces"

In a Nut Shell: Surface integrals also appear in vector form

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \qquad \textbf{Note:} \quad \mathbf{dS} = \mathbf{n}\, dS$$

They involve the "dot product" of a vector function, $\mathbf{F} = \mathbf{F}(x,y,z)$ with $\mathbf{n}$, the unit normal to the surface, S. In such cases the unit normal to the surface may point out (say the top of the surface) or may point in (say the bottom of the surface).

Such a surface, S, is said to be **"orientable"**. The direction of the unit vector, $\mathbf{n}$, establishes the orientation of the surface.

Definitions:

$\mathbf{F}$ = vector field (one example is fluid velocity giving rise to flux across a surface)
$\mathbf{dS}$ element of oriented surface S where $\mathbf{dS} = \mathbf{n}\, dS$
$\mathbf{n}$ = unit vector normal to oriented surface

as before the unit vector comes from the cross product $\mathbf{n} = (\mathbf{r}_u \times \mathbf{r}_v) / |\mathbf{r}_u \times \mathbf{r}_v|$
where $\mathbf{r}$ is the position vector to the surface, S, given by

$$\mathbf{r} = \langle x, y, z \rangle = \langle x(u,v), y(u,v), z(u,v) \rangle = \mathbf{r}(u,v)$$

"Oriented" Surface Integral:

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) / |\mathbf{r}_u \times \mathbf{r}_v|\, dS$$

and recall $dS = |\mathbf{r}_u \times \mathbf{r}_v| dA$ where $|\mathbf{r}_u \times \mathbf{r}_v|$ transforms the element of area, dS, on the surface, S, to the element of area, dA, on the uv-surface in the domain, D.

So the surface integral becomes **note:** $dS = |\mathbf{r}_u \times \mathbf{r}_v| dA$

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) / |\mathbf{r}_u \times \mathbf{r}_v|\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA$$

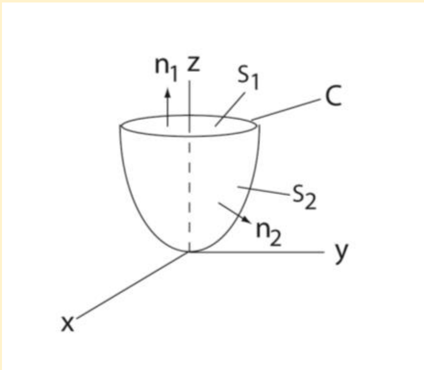
Note that the dot product, $\mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)$, is a scalar function and therefore it is a scalar field. So the same two approaches, **the direct approach and the transformation approach, still apply** to evaluate the scalar form of surface integrals starting with this vector form of surface integral.

Key point, you should determine from the statement of the problem whether the unit normal points above or below (in or out, up or down) the surface, S.

Note: S is the total surface and may consist of more than one individual surface as illustrated in the figure below.

In this situation imagine flux passing across surfaces,

S_1 (a plane) and S_2 (a paraboloid).



$$I_S = \iint_S \mathbf{F} \bullet d\mathbf{S} = \iint_S \mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) / |\mathbf{r_u} \times \mathbf{r_v}| \, dS$$

Now $dS = |\mathbf{r_u} \times \mathbf{r_v}| \, dA$ so $I_S = \iint_D \mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) \, dA$

Recall that when the surface, S, is described as follows:

$z = g(x,y)$ (recall let $u = x$ and $v = y$)

the position vector to an arbitrary point on S is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + g(x,y)\mathbf{k}$

Then one can use x and y rather than u and v as the parameters describing the surface, S.

So the normal vector to S is $\mathbf{r_u} \times \mathbf{r_v} = \mathbf{r}_x \times \mathbf{r}_y$ and

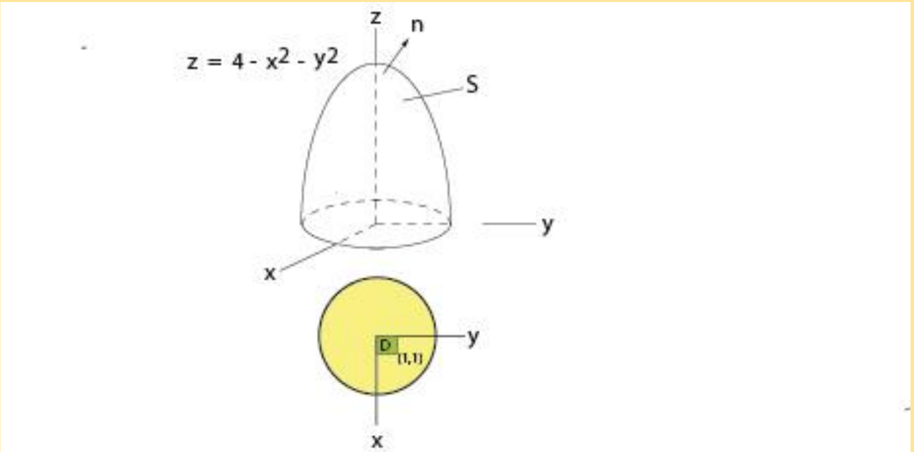
$$I_S = \iint_D \mathbf{F} \bullet (\mathbf{r_x} \times \mathbf{r_v}) \, dA$$

Example: Evaluate the surface integral $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$

where $\mathbf{F} = < xy,\ yz\ ,\ xz >$

and S is the part of the paraboloid $z = 4 - x^2 - y^2$ (not shown) that lies above

the square $0 \leq x \leq 1, 0 \leq y \leq 1$ (domain D) and has an upward orientation.



Step 1: Express **F**(x,y,z) in terms of the independent variables (in this case) x and y.

So in this example $\mathbf{F}(x,y,z(x,y)) = < xy, y(4 - x^2 - y^2), x(4 - x^2 - y^2) >$

Step 2: Write the surface area element, dS in terms of dA. $dS = |\mathbf{N}|\, dA$

In this example the projection of dS is a circle of radius 1 in the xy-plane.

Step 3: Calculate **N** : $\mathbf{N}(x,y) = \mathbf{r}_x \times \mathbf{r}_y$ where $\mathbf{r} = < x,\ y,\ z >$ so

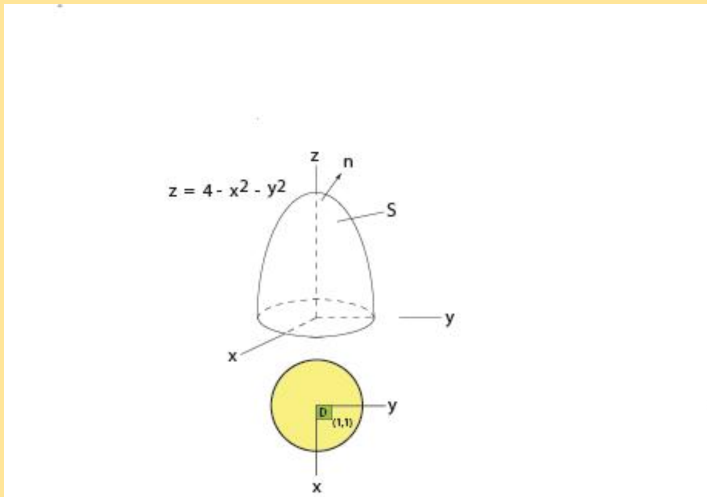
$\mathbf{r} = < x,\ y,\ (4 - x^2 - y^2) >$ so $\mathbf{r}_x = < 1, 0, -2x >$ and $\mathbf{r}_y = < 0, 1, -2y >$

so taking the cross product, $\mathbf{N} = < 2x\ , 2y, 1 >$ and $|\mathbf{N}| = \sqrt{(4x^2 + 4y^2 + 1)}$

$\mathbf{n} = \mathbf{N} / |\mathbf{N}| = < 2x\ , 2y, 1 > / \sqrt{(4x^2 + 4y^2 + 1)}$

Note: $\mathbf{n} = \mathbf{N} / |\mathbf{N}| = < 2x\ , 2y, 1 > / \sqrt{(4x^2 + 4y^2 + 1)}$

S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square
$0 \leq x \leq 1,\ 0 \leq y \leq 1$ and has **upward orientation**. See figure below.



Proceed with calculation of the surface integral:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA \qquad \text{where } dA = dx\, dy$$

Here $\mathbf{F}(x, y, z) = \langle xy, yz, zx \rangle$ note upward orientation

Now $\mathbf{r} = \langle x, y, 4 - x^2 - y^2 \rangle$ so $\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle$

So $\mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = 2x^2 + 2y^2z + xz$ (scalar function)

On the surface, S, $z = 4 - x^2 - y^2$ So the integral becomes

$$\int_{x=0}^{x=1} \int_{y=0}^{y=1} (2x^2 + 8y^2 - 2x^2y^2 - 2y^4 + 4x - x^3 - xy^2)\, dx\, dy = 713/180$$