Centroids/Average Value of a Function

Key Concept: The centroid of an area is the idealized location where all of the area can be thought to be concentrated.

In a Nutshell: Apply the principle of first moments to calculate the centroid of areas in a plane. To calculate the x-coordinate of the centroid:

$$A\, x_c = \int x\, dA$$

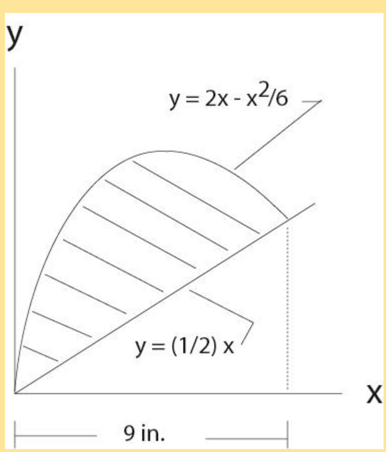
Likewise, to calculate the y-coordinate of the centroid:

$$A\, y_c = \int y\, dA$$

For Areas:
A is the total area in the xy-plane
x is the x-coordinate to the centroid of the element of area, dA
y is the y-coordinate to the centroid of the element of area, dA
x_c is the x-coordinate of the centroid
y_c is the y-coordinate of the centroid

To illustrate: Pick up a yard stick. To "balance" it at its centroid, position your finger 18 inches from each end. Equal amounts of weight on each side result in the balance.

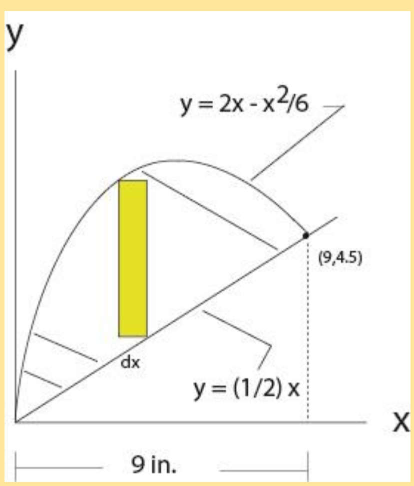
Example:
Find the x and y-coordinates of the centroid for the shaded area bounded by the curve $y = 2x - x^2/6$ and $y = (1/2)x$ shown in the figure below.



Strategy:

1.	Use the principle of first moments. See the figure below. $x_c A = \int x\, dA \qquad dA = \int [y_{upper} - y_{lower}]\, dx$ $y_c A = \int [(y_{upper} + y_{lower})/2]\ [y_{upper} - y_{lower}]\, dx$ $y_c A = \int [y^2_{upper} - y^2_{lower}](1/2)\, dx$ where $y_{upper} = 2x - x^2/6$ and $y_{lower} = (1/2)\, x$
2.	Show the element of area, dA and use it to calculate A the total area. $A = \int [y_{upper} - y_{lower}]\, dx$
3.	Determine the limits of integration.
4.	Perform the integrations and use it to solve for $x_c = \int x\, [y_{upper} - y_{lower}]\, dx) / A$ $y_c = \int [(y^2_{upper} - y^2_{lower})/2]\, dx / A$

$y_{upper} = 2x - x^2/6$ and $y_{lower} = (1/2)\, x$



The element of area, dA, = [$y_{upper} - y_{lower}$] dx

The limits of integration in the x-direction are from x = 0 to x = 9 . Calculate the total area, A.

$$A = \int_{x=0}^{x=9} \left(2x - x^2/6 - ½ x \right) dx$$

The result is A = 81/4 in^2 = 20.25 in^2

Next calculate $\int x\,dA = \int x\,[2x - x^2/6 - ½ x]\,dx = \int_{x=0}^{x=9} [2x^2 - x3/6 - ½ x^2]\,dx = 91.125$

$$x_c = 91.125/\,20.25 = 4.5 \text{ in}$$

Next calculate:

$$y_c\,A = \int [y^2_{upper}/2 - y^2_{\,lower}/2]\,dx$$

$$y_c\,A = \int_{x=0}^{x=9} [\,(2x - x^2/6)^2 - (½ x)^2\,]/2\ dx$$

$$y_c\,A = \int_{x=0}^{x=9} [\,15/8\,x^2 + x^4/72 - x^3/3\,]\ dx = 5/8\,x^3 + x^5/360 - x^4/12\ \Big|_0^9$$

$$y_{cg}\,A = [\,(9)^3 / 2\,][\,5/4 + 81/180 - 3/2\,] = 81(36)/40\ \text{in}^3$$

So y_{cg} = [81 (36)/40] / (81/4) = 36/10 = 3.60 in. (result)

Average Value of a Function

In a Nut Shell: The average value of a function, f_{ave}, over an interval from x = a to x = b is defined as follows:

$$f_{ave} = [1/(b\text{-}a)] \int_{x=a}^{x=b} f(x)\,dx$$

Example: Find the average value of f(x) = sin^2(x) over the interval [0,π].

$$f_{ave} = (1/\pi) \int_{x=0}^{\pi} \sin^2(x)\,dx$$

Use the trig substitution $\sin^2(x) = [1 - \cos(2x)] / 2$

$$f_{ave} = (1/\pi) \int_{x=0}^{x=\pi} [1 - \cos(2x)] / 2 \, dx = (1/\pi) [x - (\sin 2x)/2]/2] \Big|_{0}^{\pi}$$

$$f_{ave} = (1/\pi) [\pi/2 - \sin 2\pi /4] = 1/2$$