Line Integrals

In a Nut Shell: The definite integral you studied in integral calculus

$$\int_a^b f(x)\,dx$$ can be thought of as an integral of f(x) along the x-axis.

Similarly, an integral could be evaluated along a curve in a plane or a curve in space.
Such integrals are called **"line integrals".**

Suppose f(x, y, z) is a smooth curve in space defined by the parameter t as follows

$$x = x(t), \quad y = y(t), \quad z = z(t)$$

Now by the Pythagorean theorem $ds = \sqrt{[(dx)^2 + (dy)^2 + (dz)^2]}$

where ds represents the differential (arc) length along the curve , s , in space.

So $$\int_a^b f(x, y, z)\,ds = \int_a^b f(x, y, z)\,(ds/dt)\,dt =$$ line integral of f(x,y,z) along the curve, s

$$\int_a^b f(x, y, z)\,\sqrt{[(dx/dt)^2 + (dy/dt)^2 + (dz/dt)^2]}\,dt$$

For functions in a plane the vector field is $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$
and the vector element along the curve, C, in the plane is $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$

In this case the dot product

$\mathbf{F}(x, y) \cdot d\mathbf{r}$ yields the following line integral along C in the plane

$$\int_C P(x, y)\,dx + \int_C Q(x, y)\,dy$$

The line integral can also be expressed in terms of each coordinate variable (x, y, z).
Suppose **F**(x, y, z) is a vector field defined as

$$\mathbf{F}(x, y, z) = P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}$$

where P(x, y, z), Q(x, y, z), and R(x, y, z) are continuous functions of x, y, and z
and d**r** is a vector element of length along the curve C in space

here $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ Then the dot product

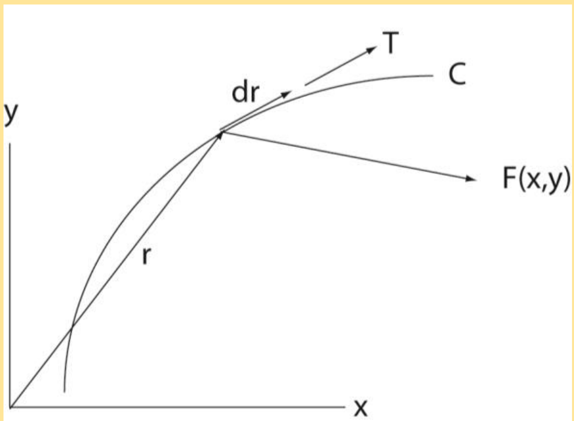
$\mathbf{F}(x, y, z) \cdot d\mathbf{r}$ yields the following line integral along the curve C in space

$$\int_C P(x, y, z)\,dx + \int_C Q(x, y, z)\,dy + \int_C R(x, y, z)\,dz$$

Both dot products discussed here $\mathbf{F}(x, y) \cdot d\mathbf{r}(x, y)$ and $\mathbf{F}(x, y, z) \cdot d\mathbf{r}(x, y, z)$

appear in engineering as the incremental work of the “force” **F** along the path, C. i.e.

Suppose **F** is a force acting on a particle in the x-y plane and **r** is the position vector from the origin to the particle.



Let the particle move an amount ds along the curve, C, in the x-y plane under the influence of the force **F .** The change in the position vector along the curve (tangent to C) is d**r** . Then the incremental work, dW, done on the particle by the force **F** is the dot product of **F** and d**r** . Thus the line integral along C gives the work, W, done by the force **F** acting on the particle as it moves along C.

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int \mathbf{F} \cdot (d\mathbf{r}/dt)\, dt \quad \text{and} \quad d\mathbf{r} = \mathbf{T}\, ds,$$ **T** is the unit tangential vector to the curve, C

It is possible that the line integral of the function, **F**, is independent of the curve (path), C,

in the x-y plane. This situation occurs when the force, **F**,

$$\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

is **conservative** In such a case, the curl of **F** must be zero.

curl of **F** = $\partial Q/\partial x - \partial P/\partial y = 0$ if the force is conservative

So $\qquad \partial P/\partial y = \partial Q/\partial x$

And the (vector field) force, **F** , can be expressed in terms of the gradient of a potential function (scalar function, φ)

$\varphi(x,y)$. i.e.

$$\mathbf{F} = \text{grad}\,(\varphi)$$

Suppose you were to evaluate the line integral

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy$$

where the path (curve) C is somewhat complicated. Then you might first check to see if the "force" (vector field) is conservative.

$$\mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

i.e. Does $\partial P/\partial y = \partial Q/\partial x$? If so, the force, **F**, is conservative.

Then the line integral, $\int_C \mathbf{F} \cdot d\mathbf{r}$, is independent of its path and you can simplify the

calculation by selecting an easier path.

Example: Evaluate the line integral for $P(x,y) = y^2$ and $Q(x,y) = x$ along the curve, C,

given by $x = y^3$ from (-1, -1) to (1, 1).

$$I = \int_C y^2\, dx + \int_C x dy$$ For the given curve, C, $dx = 3y^2\, dy$

So the line integral becomes $I = \int_{-1}^{1} y^2\, 3y^2\, dy + \int_{-1}^{1} y^3\, dy$

$$I = (3/5)y^5 + (1/4)\, y^4 \Big|_{-1}^{1} = [(3/5) +)] – [-(3/5) + (1/4)] = 6/5$$

Example: Evaluate the line integral $I = \int_C f(x,y,z)\, ds$ for $0 \le t \le 1$ where

$f(x,y,z) = 2x + 9xy$ along the curve, C, given by $x = t$, $y = t^2$, $z = t^3$

$I = \int f(x,y,z)\, ds = \int f(x,y,z)\, (ds/dt)\, dt$, $ds/dt = \sqrt{[\, dx/dt)^2 + (dy/dt)^2 + dz/dt)^2]}dt$

$$I = \int_0^1 [2t + 9\, t\, t^2\,]\ \sqrt{[1 + (2t)^2 + (3t^2)^2]}\, dt = \int_0^1 [2t + 9t^3]\ \sqrt{[1 + 4t^2 + 9t^4]}\, dt$$

Let $w = 1 + 4t^2 + 9t^4]$, then $dw = 4\, (2t + 9t^3)\, dt$

or $(2t + 9t^3)\, dt = (1/4)\, dw$ so the integral becomes

$$I = \int_1^{14} (1/4)\, w^{1/2}\, dw = (1/4)[(2/3)w^{3/2})] \Big|_1^{14} = (1/6)[14\sqrt{14} - 1]$$

Example: Determine if the force, **F**, is conservative where $\mathbf{F} = 2x\,e^y\,\mathbf{i} + x^2\,e^y\,\mathbf{j}$

$\mathbf{F} = P(x,y)\,i + Q(x,y)\,\mathbf{j}$ Now calculate $\partial P/\partial y$ and $\partial Q/\partial x$.

$\partial P/\partial y = 2x\,e^y$ and $\partial Q/\partial x = 2x\,e^y$ Since $\partial P/\partial y = \partial Q/\partial x$ **F** is conservative.

Next determine the value of the line integral

$I = \int 2x\,e^y\,dx + x^2\,e^y\,dy$ from (0,0) to (1,-1)

Note: The value of the line integral should not depend on the path.

Pick an easy path as follows: Integrate along the x-axis from $x = 0$ to $x = 1$ ($y = 0$)

Then integrate along the line $x = 1$ from $y = 0$ to $y = -1$. With this path the

integral simplifies to the following:

$$I = \int_0^1 2x\,e^0\,dx + \int_0^{-1} 1^2\,e^y\,dy = x^2\Big|_0^1 + e^y\Big|_0^{-1} = 1 + (e^{-1} - 1) = e^{-1}$$

Example: Since the force, **F**, in the previous example is conservative, it must be equal to

the gradient of a potential function, φ . So next, let's find this potential function.

$\mathbf{F} = \text{grad}\,\varphi = \partial\varphi/\partial x\,\mathbf{i} + \partial\varphi/\partial y\,\mathbf{j}$

But $\mathbf{F} = 2x\,e^y\,\mathbf{i} + x^2\,e^y\,\mathbf{j}$ so

$\partial\varphi/\partial x = 2x\,e^y$ and $\partial\varphi/\partial y = x^2\,e^y$

Integrate $\partial\varphi/\partial x$ with respect to x to obtain $\varphi = x^2\,e^y + f(y)$

Then take the derivative with respect to y to obtain $\partial\varphi/\partial y = x^2\,e^y + df(y)/dy$

Next equate this result to $\partial\varphi/\partial y$ from above to obtain

$$x^2\,e^y = x^2\,e^y + df(y)/dy$$

Thus $df/dy = 0$ or $f(y) = C = \text{constant}$

Thus the potential function, $\varphi(x, y) = x^2\,e^y + C$ (result)

Example: For the previous example use the gradient of the potential function that was

found to recalculate the line integral, $\int \mathbf{F} \cdot d\mathbf{r}$.

Recall the potential function was found to be $\varphi(x, y) = x^2 e^y + C$

$\mathbf{F} = \text{grad}\,\varphi = \partial\varphi/\partial x\,\mathbf{i} + \partial\varphi/\partial y\,\mathbf{j}$

So $\mathbf{F}\cdot d\mathbf{r} = \text{grad}\,\varphi \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) =$

$[\partial\varphi/\partial x\,\mathbf{i} + \partial\varphi/\partial y\,\mathbf{j}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) = (\partial\varphi/\partial x)dx + (\partial\varphi/\partial y)dy = d\varphi$

So $\int \mathbf{F}\cdot d\mathbf{r} = \int d\varphi$.

So the line integral becomes $\int d\varphi = \varphi|_{(1,-1)} - \varphi|_{(0,0)}$ where $\varphi|_{(0,0)} = C$

Thus $I = (1^2)(e^{-1}) + C - C = e^{-1}$ (Same result as before for line integral!)