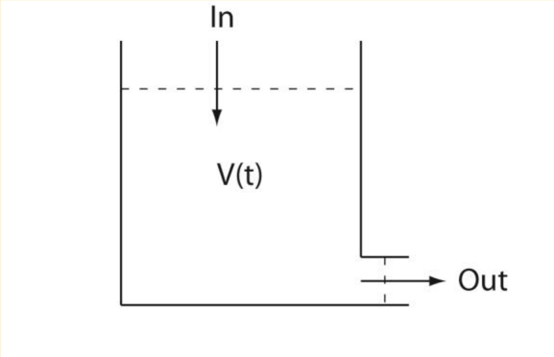
Related Rate Applications

In a Nut Shell: There are a number of related rated applications from a general family of linear, first-order differential equations that are either separable or where an integrating factor can be used to obtain a solution. One common application is the changing concentration of a solute in a tank with inflow and outflow. Another is the increase in a savings account related to deposits. First consider flow into and exiting a tank as shown in the figure below.

Applications involving changes of solute in a Tank



Flow of Solute entering and exiting a tank

Basic Differential Equation governing Related Rate for inflow and outflow

dy/dt = Rate of inflow of solute - Rate of outflow of solute = rate of change of solute in the tank

$y(t)$ = amount of solute in the tank at any time t
dy/dt = rate of change of solute in the tank at any time t
$V(t)$ = volume of substance within tank at any time t

where:

Rate of inflow = $r_i C_i$, where r_i = rate of flow in i.e. Liters per minute
C_i = concentration of solute entering in grams/liter

Rate of outflow = $r_o C_o$, where r_o = rate of flow out i.e. Liters per minute
C_o = concentration of solute exiting in grams/liter

Example: The tank shown below contains 200 L of a salt solution with a concentration of 1 gram/liter. The tank is subsequently rinsed with fresh water flowing in at a rate of 2 liters/min. The solution in the tank is continuously stirred providing complete mixing while flowing out of the tank at the same rate. Determine the time elapsed before the concentration of salt in the tank reaches 10 % of its original value.

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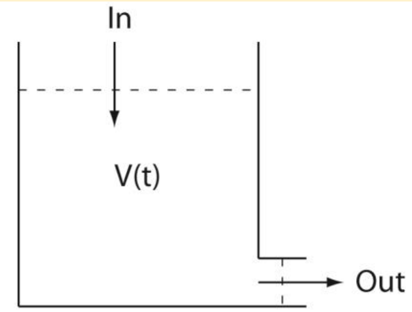


Flow of Solute entering and exiting a tank

dy/dt = Rate of inflow of solute - Rate of outflow of solute = Rate of change of solute in the tank

$dy/dt = r_i C_i - r_o C_o$ $y(0) = 200$ grams of salt

For given data:

Rate of inflow = $r_i C_i$, where r_i = rate of flow in = 2 liters/min
C_i = concentration of salt entering in grams/liter = 0 (fresh water)

Rate of outflow = $r_o C_o$, where r_o = rate of flow out = 2 liters/ min
C_o = concentration of solute exiting in grams/liter = $y(t)/V(t)$
= 2/200 grams/min Note: Volume at any time = 200 L

So the governing d.e. is: $dy/dt = -(1/100)\, y(t)$ with the initial condition $y(0) = 200$

Separate variables: $dy/y = -(1/100)\, dt$ or $\ln(y) = -(1/100)\, t + C_1$

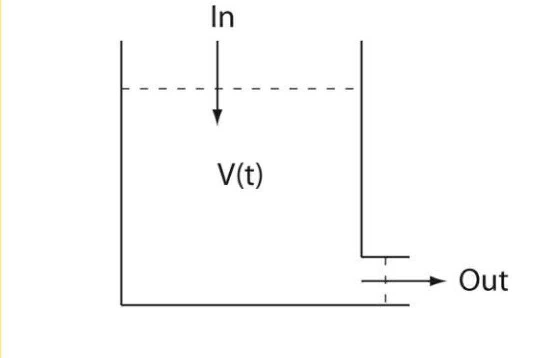
Solve for y(t) $y(t) = C \exp[(-1/100)\, t]$ and $y(0) = 200 = C$

So $y(t) = 200 \exp[(-1/100)\, t]$ and when $y(t^*) = 20$, $1/10 = \exp[(-1/100)\, t^*]$

$(-1/100)\, t^* = \ln(1/10)$ and $t^* = 100 \ln(10)$ minutes (result)

Example: The tank shown below starts with 120 L of pure water. Entering the tank at a rate of 4 liters/minute is a mixture of X grams/liter of salt. with a concentration of 1 gram/liter. The solution in the tank is continuously stirred providing complete mixing while flowing out of the tank at the same rate. Determine an expression for the amount of salt in the tank at any time t in terms of X. What happens to the amount of salt in the tank after a very long period of time?

Applications involving changes of solute in a Tank



Flow of Solute entering and exiting a tank

dy/dt = Rate of inflow of solute - Rate of outflow of solute = Rate of change of solute in the tank

$dy/dt = r_i C_i - r_o C_o$ $y(0) = 0$ grams of salt

Rate of inflow = $r_i C_i$, where r_i = rate of flow in = 4 liters/min
C_i = concentration of salt entering in grams/liter = X (fresh water)

Rate of outflow = $r_o C_o$, where r_o = rate of flow out = 4 liters/ min
C_o = concentration of solute exiting in grams/liter = $y(t)/V(t)$
= $(4/120)\, y$ grams/min   Note: Volume at any time = 120 L

So the governing d.e. is: $dy/dt = 4X - (1/30)\, y(t)$ with the initial condition $y(0) = 0$

$dy/dt + (1/30)\, y(t) = 4X$ Apply the integrating factor is $e^{(1/30)t}$

The d.e. becomes: $d/dt\, [e^{(1/30)t}\, y] = 4X\, e^{(1/30)t}$ or $y = 120\,X + C_1\, e^{-(1/30)t}$

Solve for y(t) $y(0) = 0$ So $C_1 = -120\,X$ and $y(t) = 120\,X\,[\,1 - e^{-(1/30)t}\,]$ (result)

So after a very long time $y(t \to \infty) = 120\,X$ grams of salt in the tank.

Example involving a Financial Application

Lucy has no money available for initial capital but decides to invest k dollars per year at an annual rate of return, r. Assume that the investments are made continuously and that the rate of return, r, is also compounded continuously.

a. Find the sum, S(t), in Lucy's retirement account accumulated at any time, t.
b. Suppose the annual rate of return, r = 7.5 %. Find how much Lucy must invest each year if she will have $1 million dollars available to her for retirement in 40 years.
c. If Lucy invests k = $2000/year, find the return rate, r, that must be obtained to have $1 million dollars available in 40 years.

Strategy: Apply the governing first order differential equation:

$$dS/dt = rS + k$$

In other words, the rate of change of the amount, $S(t)$, in Lucy's account equals the annual rate times the current amount in her account plus any amount, k, invested by Lucy each year.

Nest, rewrite the d.e. as $dS/dt - rS = k$ (Note the integrating factor, e^{-rt})

So the general solution becomes $S(t) = C e^{rt} - k/r$

with the initial condition that $S(0) = S_o$

Thus the investment relation becomes:

$$S(t) = S_o e^{rt} + (k/r)(e^{rt} - 1)$$

where $S(t)$ = the amount in Lucy's retirement account in any year, t
S_o = the initial capital investment which in this case is zero
r = annual rate of return
k = amount invested each year

For $S_o = 0$, $S(t) = (k/r)(e^{rt} - 1)$ (result for part a)

For part b: If $r = 7.5\%$, then $1{,}000{,}000 = (k/0.075)(e^{(0.075)(40)} - 1)$

Solving for k yields $k = \$3929.67$ (result for part b)

If $k = \$2000$/year, then $1{,}000{,}000 = (2000/r)(e^{r(40)} - 1)$

which yields $500 r = e^{r(40)} - 1$

Solve by successive iterations. The result is $r = 0.0977$ or $r = 9.77\%$ (result for part c)