The Chain Rule for Functions of Several Variables

In a Nut Shell: Recall that the chain rule for a function, y, of one independent variable, x, was very important in that it enabled you to replace the derivative of a complicated function with a product of simpler derivatives, called the "chain" rule.

Recall the Chain Rule: If $u = u(x)$ and $x = x(t)$, then $du/dt = [du/dx][dx/dt]$

In a Nut Shell: The chain rule extends to functions of more than one independent variable.

Now consider a function, $w(x,y)$ (which has continuous, first-order derivatives) and that $x = g(t)$ and $y = h(t)$ are differentiable functions. Then w is a differentiable function of t .

Terminology:

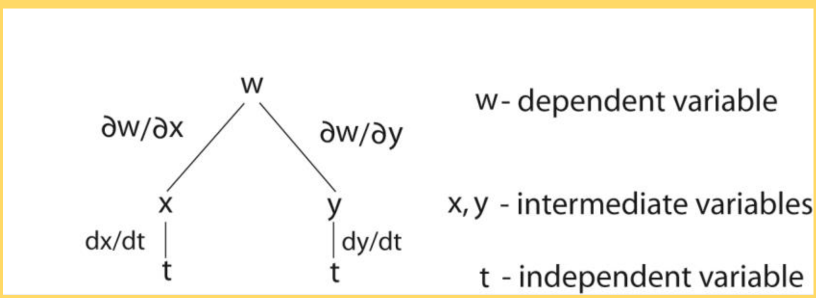
w is the dependent variable, x and y are intermediate variables

and t is the independent variable

Suppose $w = w(x,y)$ has continuous first-order derivatives. Then

$$dw/dt = [\partial w / \partial x]\, dx/dt + [\partial w / \partial y]\, dy/dt \qquad \text{(chain rule)}$$

It can be helpful to construct a **"tree diagram"** to illustrate the chain rule structure.



Tree diagram for $w = w(x,y)$ where $x = x(t)$ and $y = y(t)$.

It follows then that the chain rule for dw/dt is given as above by the expression:

$$dw/dt = [\partial w/ \partial x]\, dx/dt + [\partial w/ \partial y]\, dy/dt$$

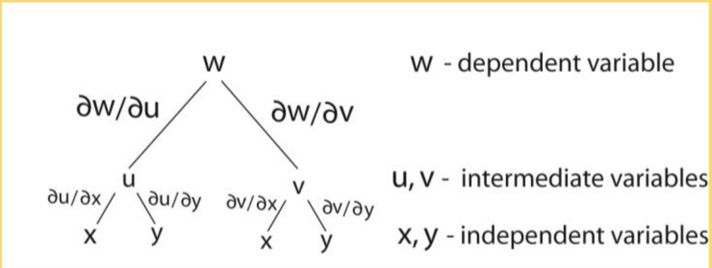
The concept of the tree diagram can be extended to many other cases. i.e. Consider the case where the dependent variable, w, has two intermediate variables, u and v, and two independent variables, x and y.

$$w = w(u, v) \text{ where } u = u(x,y),\ v = v(x,y) \qquad \text{where}$$

w is the dependent variable, u and v are intermediate variables,

and x and y are the independent variables

The **tree structure** for the dependent variable, w, is as follows
:



So the two partial derivatives for w(x,y) are as follows: **(note "product" of partials)**

$\partial w/\partial x = [\partial w/ \partial u]\, \partial u/\partial x + [\partial w/ \partial v]\, \partial v/\partial x$

and $\partial w/\partial y = [\partial w/ \partial u]\, \partial u/\partial y + [\partial w/ \partial v]\, \partial v/\partial y$

Now consider the case where the dependent variable, w, has three intermediate

variables and two independent variables.

$w = w(x, y, z)$ where $x = x(u,v)$, $y = y(u,v)$, and $z = z(u,v)$

w is the dependent variable, x, y, and z are intermediate variables

and u and v are the independent variables **(note "product" of partials)**

So $\partial w/\partial u = [\partial w/ \partial x]\, \partial x/\partial u + [\partial w/ \partial y]\, \partial y/\partial u + [\partial w/ \partial z]\, \partial z/\partial u$

and $\partial w/\partial v = [\partial w/ \partial x]\, \partial x/\partial v + [\partial w/ \partial y]\, \partial y/\partial v + [\partial w/ \partial z]\, \partial z/\partial v$

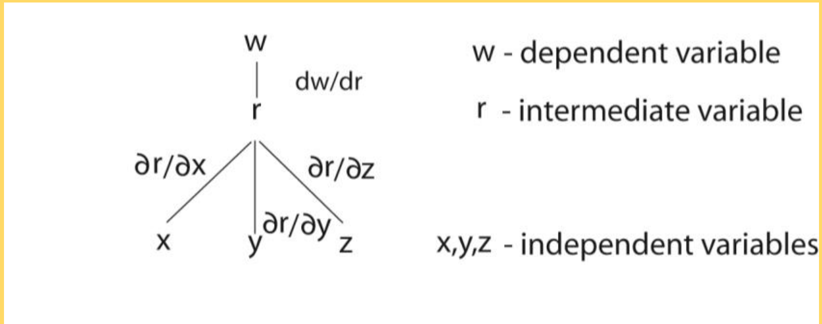
Example: Let $w = f(r)$ where $r = r(x, y, z) = \sqrt{[x^2 + y^2 + z^2]}$

Here w is the dependent variable, r is the intermediate variable, and

x, y, and z are the independent variables

Show that $\partial^2 w/ \partial x^2 + \partial^2 w/ \partial y^2 + \partial^2 w/ \partial z^2 = d^2 w/ dr^2 + [2/r][dw/ dr]$

The tree structure for this problem is as follows:



Start with the first derivative.

$\partial w/\partial x = [dw/dr]\,[\partial r/\partial x] = dw/dr\,(1/2)\,[x^2 + y^2 + z^2]^{-1/2}\,[2x]$

So $\partial w/\partial x = [\,x/\sqrt{[x^2 + y^2 + z^2]}\,]\ dw/dr = (x/r)\ dw/dr$

Similarly, $\partial w/\partial y = [\,y/r\,]\ dw/dr$ and $\partial w/\partial z = [\,z/r\,]\ dw/dr$

Next find the second derivatives. **This is the hard part of the calculation.** The second derivative is just the derivative of the first derivative. Use the result for $\partial w/\partial x$.

$\partial^2 w/\partial x^2 = \partial/\partial x\,[\partial w/\partial x] = \partial/\partial x\,\{\,dw/dr\ [\partial r/\partial x]\,\}$

Next use the product rule to obtain

$\partial^2 w/\partial x^2 = \partial/\partial x\{dw/dr\}\,[\partial r/\partial x] + dw/dr\ \partial/\partial x\,\{\partial r/\partial x\}$

Next interchange the order of integration for the first term on rhs which gives

$\partial^2 w/\partial x^2 = d/dr\{\partial w/\partial x\}\,[\partial r/\partial x] + dw/dr\ \partial/\partial x\,\{\partial r/\partial x\}$

Now $\partial w/\partial x = [dw/dr]\,[\partial r/\partial x]$

Perform the differentiations on the rhs (right hand side) gives

$\partial^2 w/\partial x^2 = d^2w/dr^2\,[\partial r/\partial x]^2 + dw/dr\ [\partial^2 r/\partial x^2]$

Similar calculations and strategy occur for the second derivatives of w with respect to y and to z.

From, $\partial^2 w/\partial x^2 = d^2w/dr^2\,[\partial r/\partial x]^2\} + dw/dr\ \{\partial^2 r/\partial x^2\}$

$\partial^2 w/\partial y^2 = d^2w/dr^2\,[\partial r/\partial y]^2\} + dw/dr\ \{\partial^2 r/\partial y^2\}$

$\partial^2 w/\partial z^2 = d^2w/dr^2\,[\partial r/\partial z]^2\} + dw/dr\ \{\partial^2 r/\partial z^2\}$

Recall $r = r(x, y, z) = \sqrt{[x^2 + y^2 + z^2]}$

$$\partial r/\partial x = x / \sqrt{[x^2 + y^2 + z^2]} = x/r \quad \text{so} \quad [\partial r/ \partial x]^2 = x^2/r^2$$

$$\partial r/\partial y = y / \sqrt{[x^2 + y^2 + z^2]} = y/r \quad \text{so} \quad [\partial r/ \partial y]^2 = y^2/r^2$$

$$\partial r/\partial z = z / \sqrt{[x^2 + y^2 + z^2]} = z/r \quad \text{so} \quad [\partial r/ \partial z]^2 = z^2/r^2$$

Next find the second derivatives.

$$\partial^2 r/\partial x^2 = 1 / \sqrt{[x^2 + y^2 + z^2]} - x^2 / [x^2 + y^2 + z^2]^{3/2} = 1/r - x^2/ r^3$$

Similarly,

$$\partial^2 r/\partial y^2 = 1 / \sqrt{[x^2 + y^2 + z^2]} - y^2 / [x^2 + y^2 + z^2]^{3/2} = 1/r - y^2/ r^3$$

$$\partial^2 r/\partial z^2 = 1 / \sqrt{[x^2 + y^2 + z^2]} - z^2 / [x^2 + y^2 + z^2]^{3/2} = 1/r - z^2/ r^3$$

Put these results for the first and second derivatives into $\partial^2 w/ \partial x^2$, $\partial^2 w/ \partial y^2$, and $\partial^2 w/ \partial z^2$

$$\partial^2 w/ \partial x^2 + \partial^2 w/ \partial y^2 + \partial^2 w/ \partial z^2 = d^2w/dr^2 + \{ [2(x^2 + y^2 + z^2) / r^3] dw/dr \}$$

or $\partial^2 w/ \partial x^2 + \partial^2 w/ \partial y^2 + \partial^2 w/ \partial z^2 = d^2w/dr^2 + (2/r) dw/dr$ **which is the result**