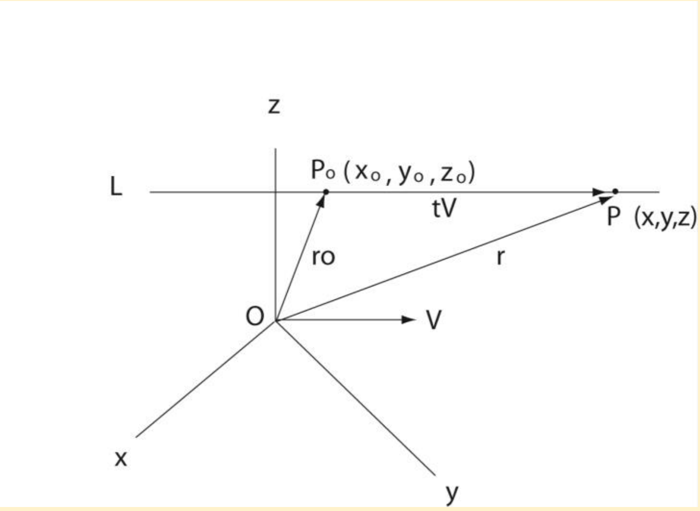
Lines and Planes in Space

In a Nut Shell: There are **three types of lines in space**. Those that are parallel, those that intersect, and those that are skew (neither parallel nor intersect). The equation of a line in space is found by vector addition.

Strategy: Let **r** = < x, y, z > be a vector from the origin, O, to an arbitrary point P(x, y, z) on a line, L, in space. (Figure below) Let $\mathbf{r_o}$ = < x_o, y_o, z_o > be a vector from the origin to the point P_o (x_o, y_o, z_o) on the same line, L. Let **V** be a vector < a, b, c > parallel to the line L. Let t be a constant parameter.

Now let t**V** be a vector along (or parallel to line L) such that

$t\mathbf{V} = ta\,\mathbf{i} + tb\,\mathbf{j} + tc\,\mathbf{k}$ is the vector from $\mathbf{r_o}$ to **r**.



Then by vector addition: (Key step in determining the equation for the line)

$\mathbf{r} = \mathbf{r_o} + t\mathbf{V}$ (This is the equation of the line in vector form.)

Equation of the line, L, in scalar form: $x = x_o + at$, $y = y_o + bt$, $z = z_o + ct$

The equation of the line, L, in “symmetric” form is obtained from the scalar form by solving for t. The result is:

$[x - x_o] / a = [y - y_o] / b = [z - z_o] / c = t$

Example: Determine whether the two lines L_1 and L_2 are parallel.

L_1 $x = 6 + 2t, \quad y = 5 + 2t, \quad z = 7 + 3t$

L_2 $x = 7 + 3s, \quad y = 5 + 3s, \quad z = 10 + 5s$

Note the lines in "symmetric" form are:

$$(x - 6)/2 = (y - 5)/2 = (z - 7)/3 = t$$

and $(x - 7)/3 = (y - 5)/3 = (z - 10)/5 = s$

So, the vector parallel to line L_1 is $V_1 = 2\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

and the vector parallel to line L_2 is $V_2 = 3\,\mathbf{i} + 3\,\mathbf{j} + 5\,\mathbf{k}$

Since V_1 is not a multiple of V_2 , these vectors and the lines are not parallel. (result)

Next determine if the lines L_1 and L_2 skew or intersecting.

If the lines intersect, then they must have a common point at the point of intersection.
If that is not the case, then the lines are skew.

So **assume that the two lines intersect.** Then:

$$x = 6 + 2t = 7 + 3s \qquad \text{eq (1)}$$

$$y = 5 + 2t = 5 + 3s \qquad \text{eq (2)}$$

$$z = 7 + 3t = 10 + 5s \qquad \text{eq (3)}$$

Subtract (2) from (3). The result is: $2 + t = 5 + 2s$ or $t = 3 + 2s$

So $2t = 6 + 4s$. Put this into (2) and solve for s.

$$5 + 6 + 4s = 5 + 3s, \quad \text{or} \quad s = -6$$

Then put this result into $t = 3 + 2s$ which yields $t = -9$

Now determine if these values of s and t satisfy eq (1).

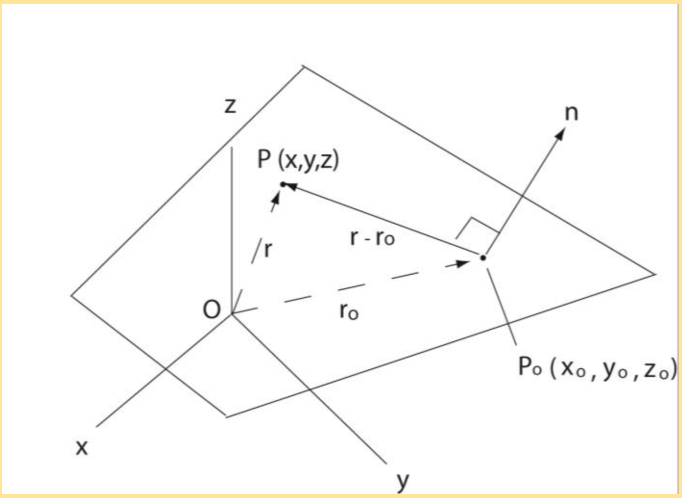
$$6 + 2(-9) =? \ 7 + 3(-6) \quad \text{which does not hold!}$$

So the lines L_1 and L_2 do not intersect. Therefore they are skew. (result)

PLANES

In a Nut Shell: The equation of a plane is found by taking the dot product of any line in the plane with its normal vector, **n** . The result is that the dot product is zero since the normal vector is perpendicular to the arbitrary line in the plane.

Strategy: Let **r** = < x, y, z > be a vector from the origin, O, to an arbitrary point P(x,y,z) in the plane. Let $\mathbf{r_o}$ = < x_0, y_0, z_0 > be a vector from the origin to the point $P_o(x_o, y_o, z_o)$ also in the plane. So the vector $\mathbf{r} - \mathbf{r_o}$ is a vector within the plane to be determined. Let < **i**, **j**, **k** > be unit vectors along the x, y, and z axes.



The dot product of the vector $\mathbf{r} - \mathbf{r_o}$ with the normal vector **n** (**n** is normal to every line in the plane) must equal zero since the vectors are perpendicular to each other.

$$(\mathbf{r} - \mathbf{r_o}) \cdot \mathbf{n} = 0$$

If $\mathbf{n} = a\,\mathbf{i} + b\,\mathbf{j} + c\,\mathbf{k}$, then the equation of the plane has the following form:

$$(x - x_o)\,a + (y - y_o)\,b + (z - z_o)\,c = 0$$

Note: The normal to a plane can be determined by taking the cross product of any two non-parallel lines (vectors) in the plane.

Further note: The angle between any two intersecting planes can be determined by finding the normal to each plane, say $\mathbf{n_1}$ **and** $\mathbf{n_2}$ **,** and then by finding the dot product of these two vectors. Let θ be the angle between $\mathbf{n_1}$ **and** $\mathbf{n_2}$

$$\cos\theta = \mathbf{n_1} \cdot \mathbf{n_2} \,/\, |\mathbf{n_1}|\ |\mathbf{n_2}|$$

Example: Determine the equation of the plane that passes through the origin, O, and contains the points P(1, 1, 1) and Q(1, -1, 3) .

Strategy: Determine the normal vector to the plane and form the dot product between this vector and an arbitrary vector in the plane.

Let $\mathbf{r}_o$ = **OP** be a vector from the origin to point P. So **OP** = **i** + **j** + **k**

Let **OQ** be a vector from the origin to point P. So **OQ** = **i** - **j** + 3 **k**

Since both vectors **OP** and **OQ** are vectors in the plane, the cross product yields the normal vector, **n**.

$$\mathbf{n} = \mathbf{OP} \times \mathbf{OQ} \quad \text{(vector cross product)}$$

$$\mathbf{n} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix}$$

where det denotes the 3x3 determinant

$$\mathbf{n} = \mathbf{i}\,(3 + 1) - \mathbf{j}\,(3 - 1) + \mathbf{k}\,(-1 - 1) = 4\,\mathbf{i} - 2\,\mathbf{j} - 2\,\mathbf{k}$$

Also let **r** be a vector to an arbitrary point (x, y, z) in the plane.

So $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

So $\mathbf{r} - \mathbf{r}_o = \mathbf{i}\,(x - 1) - \mathbf{j}\,(y - 1) + \mathbf{k}\,(z - 1)$

Note: $\mathbf{r} - \mathbf{r}_o$ is a vector in the desired plane.

and therefore, **n** and $\mathbf{r} - \mathbf{r}_o$ are perpendicular to each other so the dot product is zero.

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_o) = 0 = 4(x - 1) - 2(y - 1) - 2(z - 1)$$

or $2(x - 1) - (y - 1) - (z - 1) = 0$

$$2x - y - z = 0 \qquad \text{(result for equation of the plane)}$$