Lagrange Multipliers and Constrained Optimization

In a Nut Shell: Sometimes you wish to maximize or minimize a function subject to one or more limitations (called constraints). For example, one may wish to find the maximum volume of an object subject to a restriction on its surface area or you may wish to maximize profit subject to a restriction on available investment opportunities. The use of Lagrange multipliers offers a convenient approach. **Three cases will be discussed.**

Case 1 Suppose you wish to maximize/minimize a function with two independent variables, $f(x, y)$, subject to one constraint, $g(x, y)$.

The functions involved are: $f(x, y) = 0$ and $g(x, y) = 0$

Let $f(x,y)$ be the function to be optimized subject to the constraint relation, $g(x,y)$.

Both $f(x, y)$ and $g(x, y)$ must be continuously differentiable functions.

Then introduce an arbitrary constant, λ , called **the Lagrange multiplier** such that:

$$\text{Grad } f = \lambda \text{ Grad } g \quad \text{------------------------------------} \quad (1)$$

where Grad g means gradient of g

So $\partial f/\partial x\ \mathbf{i} + \partial f/\partial y\ \mathbf{j} = \lambda\ [\partial g/\partial x\ \mathbf{i} + \partial g/\partial y\ \mathbf{j}]$

In scalar form: $\partial f/\partial x = \lambda\ [\partial g/\partial x]$, $\partial f/\partial y = \lambda\ [\partial g/\partial y]$, and $g(x, y) = 0$

Which gives 3 equations in the 3 unknowns, x, y, and λ

Once solved, the values of x and y can be input to $f(x, y)$ to obtain the optimum.

Case 2 A similar approach applies to functions with three independent variables subject to one constraint. i.e. The result is 4 equations in the 4 unknowns, x, y, z, and λ

The functions involved are:

$f(x, y, z) = 0$ subject to the constraint $g(x, y, z) = 0$

$$\partial f/\partial x = \lambda\ [\partial g/\partial x], \quad \partial f/\partial y = \lambda\ [\partial g/\partial y],$$

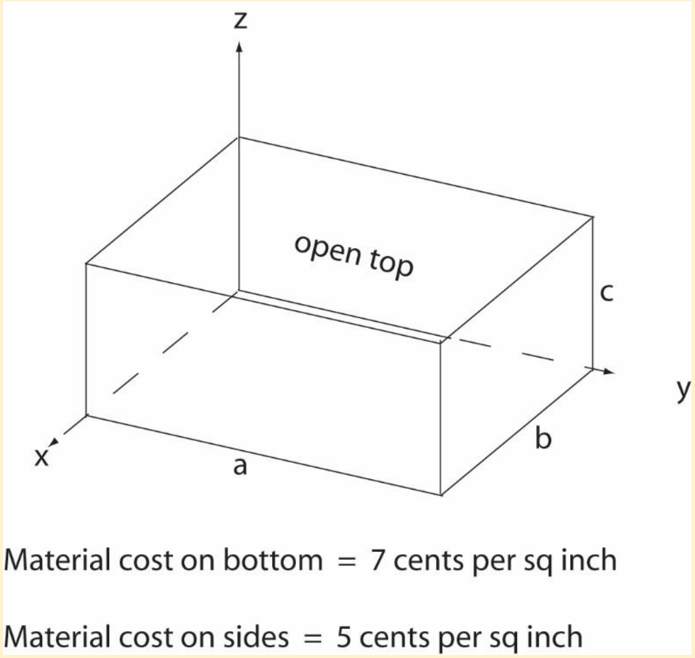
$$\partial f/\partial z = \lambda\ [\partial g/\partial z], \quad \text{and} \quad g(x, y, z) = 0$$

Case 3 A similar approach applies to one functions, f, with three independent variables, x, y, and z subject to two constraints.

The result is 5 equations in the following 5 unknowns:

x, y, z, λ_1 and λ_2 Here λ_1 and λ_2 are two Lagrange multipliers one for each constraint.

The function is: $f(x,y,z) = 0$, The constraints are: $g(x,y,z) = 0$, and $h(x,y,z) = 0$
For a maximum or for a minimum: $\partial f/\partial x = \lambda_1 [\partial g/\partial x] + \lambda_2 [\partial h/\partial x]$ $\partial f/\partial y = \lambda_1 [\partial g/\partial y] + \lambda_2 [\partial h/\partial y]$ $\partial f/\partial z = \lambda_1 [\partial g/\partial z] + \lambda_2 [\partial h/\partial z]$ with $g(x, y, z) = 0$ and $h(x, y, z) = 0$
Once one solves this system of equations for x, y, z, λ_1 and λ_2 , then the values of (x, y, z) can be input into $f(x, y, z)$ to obtain the optimal value.
Example: A rectangular box with an open top is to be designed for a volume of 700 in^3. The material for its bottom cost 7¢ /in^2 and the material for its four vertical sides costs 5¢ /in^2. Find the dimensions of the box that will minimize the cost of the materials used in constructing the box
Strategy: The first step is to determine the expression for the function and the expression for the constraint. In this case the function to be minimized is the cost function. Let the box have dimensions a, b, and c with the dimensions of the bottom of the box being a by b. The dimension of the two sides are then a by c and b by c.
The figure below shows the box with dimensions a by b by c with an open top.



Strategy: Determine the cost function, F, and the constraint function C.

$$F(a,b,c) = 7\,a\,b + 2(5)\,a\,c + 2(5)\,b\,c = 7\,a\,b + 10\,a\,c + 10\,b\,c$$

The constraint function is C(a, b, c). In this example the box is to be designed so that the volume equals 700 in^3. So the constraint function is:

$$C(a, b, c) = a\,b\,c - 700 = 0$$

Strategy: For a maximum or for a minimum set the gradient of the cost function equal to λ times the gradient of the constraint function where λ is the Lagrange Multiplier.

$$\text{Grad } F = \lambda \text{ grad } C$$

In scalar form,

$$\partial F/\partial a = \lambda\,[\partial C/\partial a] \qquad \partial F/\partial a = 7\,b + 10\,c = \lambda\,b\,c \quad \text{-------} \ (1)$$

$$\partial F/\partial b = \lambda\,[\partial C/\partial b] \qquad \partial F/\partial b = 7\,a + 10\,c = \lambda\,a\,c \quad \text{--------} \ (2)$$

$$\partial F/\partial c = \lambda\,[\partial C/\partial c] \qquad \partial F/\partial c = 10\,a + 10\,b = \lambda\,a\,b \quad \text{--------} \ (3)$$

also $F(a, b, c) = 0$ $\qquad a\,b\,c = 700$ ----------------------- (4)

The final step is to solve the 4 simultaneous equations for a, b, c, and λ .

Divide (1) by (2). $(7b + 10c) / (7a + 10c) = b / a$

Divide (2) by (3). $(7a + 10c) / (10a + 10b) = c / b$ ------------- (5)

Then cross multiply: $7ab + 10ac = 7ab + 10bc$ or $10ac = 10bc$

Since $c \neq 0$ (box must have nonzero volume) $a = b$.

Next put $a = b$ into (5) and cross multiply: $(7a + 10c)a = (10a + 10a)c$

$7a^2 + 10ac = 20ac$, or $7a^2 = 10ac$

Now $a \neq 0$, so $7a = 10c$

Finally, substitute $a = b$, $7a = 10c$ into (4) to get $(a\ a)(7/10)a = 700$

$a^3 = 1000$,

So the dimensions of the box are: $a = 10$ in and $b = 10$ in, $c = 7$ in (result)

Example: Find the points on the intersection of $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ that are the closest to and farthest from the origin. This example then contains one function (distance) subject to two constraints, the two curves forming the points of intersection.

Strategy: Use the distance (squared) of a point from the origin as the function to be minimized or maximized. Then no need to deal with square roots.

So the function, f(x,y,z), to be minimized or maximize is:

$f(x,y,z) = x^2 + y^2 + z^2$ subject to the following constraints

$g(x,y,z) = x^2 + y^2 - 1 = 0$ and $h(x,y,z) = x^2 + z^2 - 1 = 0$

Now let λ_1 and λ_2 be Lagrange multipliers. Then the optimization problem becomes

Grad f = λ_1 Grad g + λ_2 Grad h or in scalar form

$\partial f/\partial x = \lambda_1 \partial g/\partial x + \lambda_2 \partial h/\partial x$ or $2x = 2\lambda_1 x + 2\lambda_2 x$

$\partial f/\partial y = \lambda_1 \partial g/\partial y + \lambda_2 \partial h/\partial y$ or $2y = 2\lambda_1 y$

$\partial f/\partial z = \lambda_1 \partial g/\partial z + \lambda_2 \partial h/\partial z$ or $2z = 2\lambda_2 z$ and

$x^2 + y^2 - 1 = 0$ as well as $x^2 + z^2 - 1 = 0$ giving 5 equations in 5 unknowns

Simplify the equations and rewrite as follows:

$$x = \lambda_1 x + \lambda_2 x$$

$$y = \lambda_1 y$$

$$z = \lambda_2 z$$

$$x^2 + y^2 = 1$$

$$x^2 + z^2 = 1$$

Be careful in solving these equations. The last two equations of constraint are nonlinear.

Look at the case when $x \neq 0$, $y \neq 0$, and $z \neq 0$. Then one can divide by x, y, and z to give (from the first three equations)

$$1 = \lambda_1 + \lambda_2$$

$$1 = \lambda_1$$

$$1 = \lambda_2$$

which leads to the contradiction that $1 = 2$. So this case is not admissible.

Next look at the case where y and z are nonzero. Then from

$y = \lambda_1 y$ one can divide by y to give $1 = \lambda_1$

$z = \lambda_2 z$ one can divide by z to give $1 = \lambda_2$ and from

and from $x = \lambda_1 x + \lambda_2 x$ one obtains $x = 0$. Then use

$x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ to obtain $y = \pm 1$ and $z = \pm 1$

In this case $f(0, \pm 1, \pm 1) = 2$ are maxima (result)

Next look at the case where $y = 0$. Then from $x^2 + y^2 = 1$ $x = \pm 1$ and

from $x^2 + z^2 = 1$ one obtains $z = 0$

Note: The same result occurs if one considers the case where $z = 0$.

In both these cases $f(\pm 1, 0, 0) = 1$ are minima (result)