

Area Calculations/Surface Area of Revolution

In a Nut Shell: Calculation of the area under a curve or between curves is a three step process using a vertical element of area. The process for one curve is given below.

1. Given $y_1 = f_1(x)$, $y_2 = f_2(x)$, and values of x , plot the curves in the xy -plane.

2. Identify the element of area, dA , and show it on the graph. A typical element is

$$dA = (y_u - y_L) dx \quad \text{or} \quad dA = (x_R - x_L) dy$$

3. Determine the limits of integration for the area to be calculated, $a \leq x \leq b$ or $c \leq y \leq d$ by setting $y_1(x) = y_2(x)$ or $x_1(y) = x_2(y)$ then evaluate the integral:

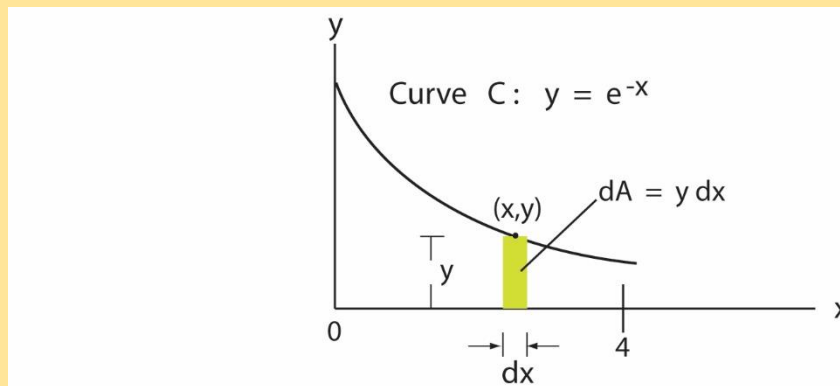
$$I = \int_a^b (y_u - y_L) dx \quad \text{or} \quad \int_c^d (x_R - x_L) dy$$

Example of one curve bounded by the x-axis and one or more vertical lines.

$$I = \int e^{-x} dx \quad \text{where } y(x) = e^{-x}$$

Let the area be bounded below by the x -axis and on each side by $[0,4]$

Steps 1 and 2: Draw curve and show the element of area.



$$\text{Here } dA = (y_u - y_L) dx, A = \int [y(x) - 0] dx \quad \text{or} \quad A = \int [e^{-x} - 0] dx$$

Step 3: Determine limits of integration. In this case the lower limit is $x = 0$ and the upper limit is $x = 4$.

Step 3: Evaluate the integral. **Note:** Area should always be positive value.

$$A = \int_0^4 e^{-x} dx = -e^{-x} \Big|_0^4 = -[e^{-4} - 1] = 1 - e^{-4}$$

Revised Strategy where the area lies between two curves in the plane is described below.

1. Given $y_1 = f_1(x)$, $y_2 = f_2(x)$, and values of x , plot the curves in the xy -plane.

2. Identify the element of area, dA , and show it on the graph.

Typical elements $dA = (y_u - y_L) dx$ or $dA = (x_R - x_L) dy$

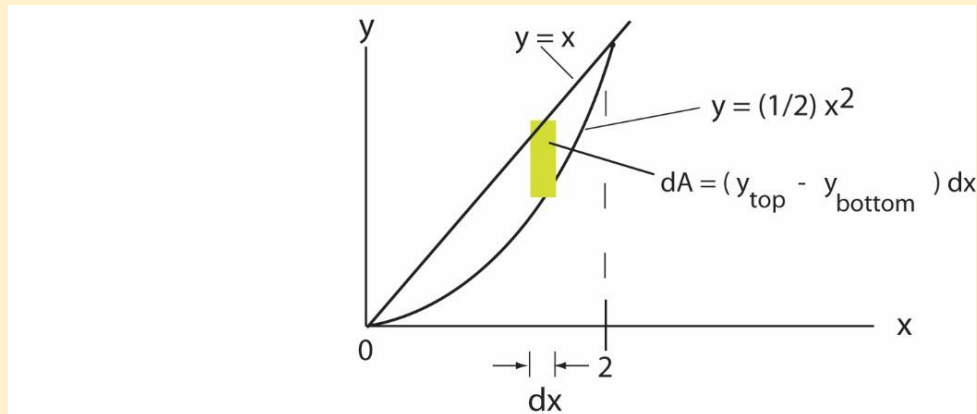
3. Determine the limits of integration $a \leq x \leq b$ or $c \leq y \leq d$ for the area to be calculated) by setting $y_1(x) = y_2(x)$ or $x_1(y) = x_2(y)$ then evaluate the integral:

$$I = \int_a^b (y_u - y_L) dx \quad \text{or} \quad \int_c^d (x_R - x_L) dy$$

Example: Find the area between the intersecting curves $y_1(x) = x$ and $y_2(x) = \frac{1}{2}x^2$

Steps 1 and 2: Draw curve and show the element of area.

$$dA = (y_{\text{top}} - y_{\text{bottom}}) dx$$



Step 3: Determine the limits of integration by finding the points of intersection of the curves $y_1(x)$ and $y_2(x)$. To do so set $y_1(x) = y_2(x)$ so $x = \frac{1}{2}x^2$ and $x(1 - 0.5x) = 0$ or $x = 0$ and $x = 2$ are the points of intersection.

Step 3: Evaluate the integral:

$$A = \int_0^2 [x - \frac{1}{2}x^2] dx = [x^2 - x^3/6]_0^2 = 2 - 4/3 = 2/3$$

More complicated situations are when the area is between two curves and a vertical line or between two curves and a horizontal line. This will not be discussed.

One can also evaluate this same example using a horizontal element of area using the steps below.

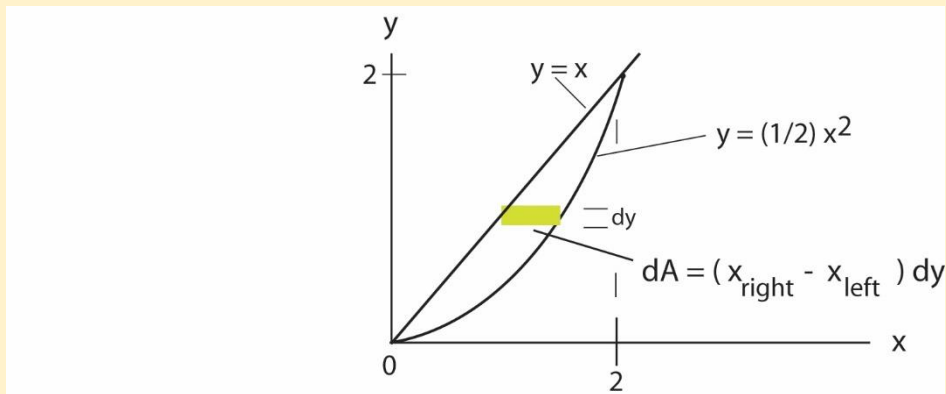
1. Given $y_1 = f_1(x)$, $y_2 = f_2(x)$, and values of x , plot the curves in the xy -plane.
2. Identify the element of area, dA , and show it on the graph shown below.
3. For a horizontal strip: $dA = (x_R - x_L) dy$

Determine the limits of integration $a \leq x \leq b$ or $c \leq y \leq d$ for the area to be calculated by setting $y_1(x) = y_2(x)$ or $x_1(y) = x_2(y)$ then evaluate the integral:

$$I = \int_a^b (y_u - y_L) dx \quad \text{or} \quad \int_c^d (x_R - x_L) dy$$

Example: Find the area between the intersecting curves $y(x) = x$ and $y(x) = \frac{1}{2}x^2$

Step 1: Draw curve and show the element of area. $dA = (x_R - x_L) dy$



Step 2: Determine the limits of integration by finding the points of intersection of the curves $y_1(x)$ and $y_2(x)$. To do so set x values equal: Here $x = y$ and $x = \sqrt{2y}$. So $y = \sqrt{2y}$ or $y^2 = 2y$. So $y = 0$ and $y = 2$ are the points of intersection.

Step 3: Evaluate the integral

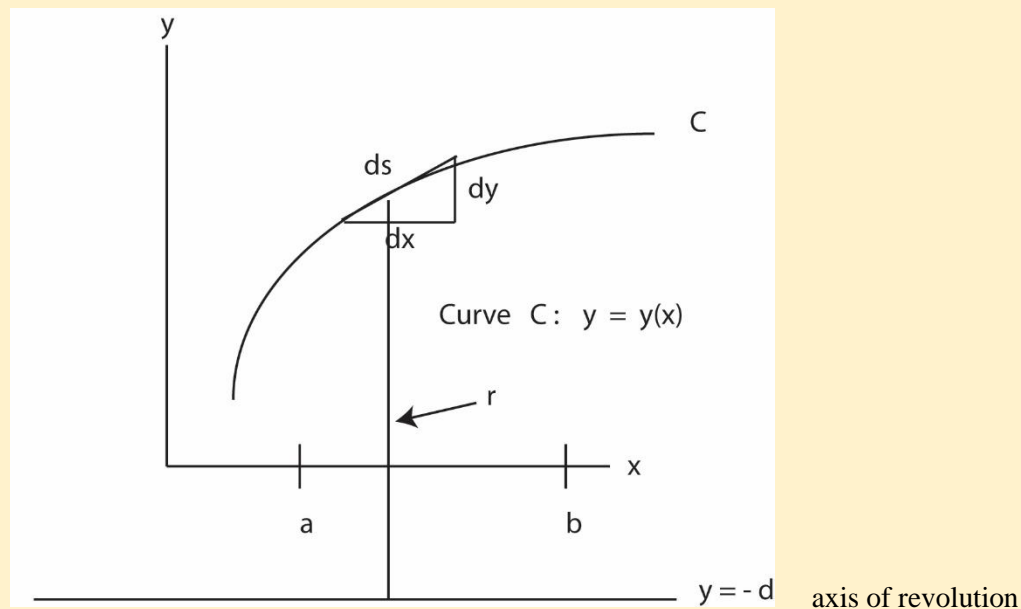
$$A = \int_0^2 [\sqrt{2y} - y] dy = \left[\frac{\sqrt{2}}{2/3} y^{3/2} - \frac{y^2}{2} \right]_0^2 = [\sqrt{2}](2/3)8^{3/2} - 2] = 2/3$$

Applications can be extended to surface areas of revolution. In this application one or more curves may be revolved about either a horizontal axis or a vertical axis. They result in a surface area of revolution. Common examples include the surface areas of a cone, of a sphere, of a hemisphere, of a cylinder, or other applications.

In a Nut Shell: Calculation of surface area of revolution, A_s , is based on the Pythagorean Theorem. The calculation typically involves three steps as follows:

Step 1 Visualize a “small” (differential) element, ds , tangent to the curve, C , at an arbitrary location (x,y) as shown in the figure below. The length ds can be calculated using the Pythagorean theorem.

$$ds^2 = dx^2 + dy^2 \quad \text{Thus} \quad ds = \sqrt{dx^2 + dy^2}$$



Step 2 For $y = y(x)$ Write ds using x as the independent variable.

$$ds = \sqrt{1 + (dy/dx)^2} dx$$

The element of surface area, $dA_s = 2 \pi r ds$, where $r = d + y$

Note: The element of area is just the circumference ($2 \pi r$) times the length of ds .

Step 3 Determine the limits of integration in order to find the total arc length. i.e.

$$a \leq x \leq b \quad \text{as shown in the figure above}$$

Perform the integration to find the total surface area of revolution. i.e.

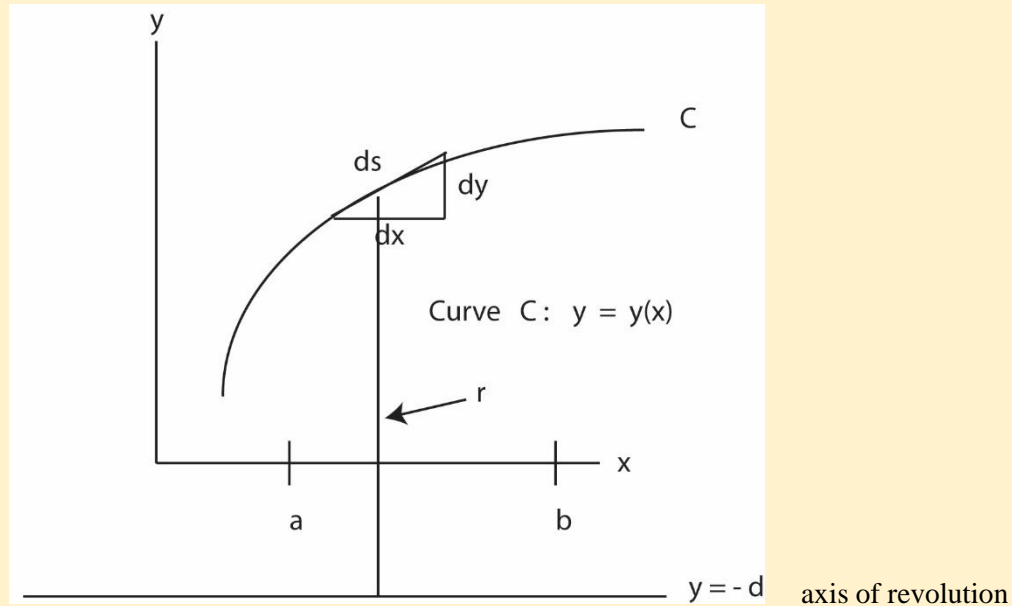
$$A_s = 2 \pi \int_a^b (d + y) \sqrt{1 + (dy/dx)^2} dx$$

Other applications include Area Calculations extended to Surface Areas of Revolution

Example: Set up the integral (but do not evaluate) the surface area for the curve C given by $y = x^{1/2}$ revolved about the line $y = -3$ for $2 \leq x \leq 6$.

Step 1 Visualize a “small” (differential) element, ds , tangent to the curve, C, at an arbitrary location (x,y) . The length ds can be calculated using the Pythagorean theorem.

$$ds^2 = dx^2 + dy^2 \quad \text{Thus} \quad ds = \sqrt{dx^2 + dy^2}$$



Step 2 For $y = y(x)$ Write ds using x as the independent variable.

$$ds = [\sqrt{1 + (dy/dx)^2}] dx \quad \text{here} \quad dy/dx = \frac{1}{2} x^{-1/2}$$

The element of surface area, $dA_s = 2 \pi r ds$, where $r = 3 + y = 3 + x^{1/2}$

Step 3 Determine the limits of integration in order to find the total arc length. i.e.

The domain of the function is: $2 \leq x \leq 6$

Perform the integration to find the total surface area of revolution. i.e.

$$A_s = 2 \pi \int_2^6 (3 + x^{1/2}) [\sqrt{1 + 1/4x}] dx$$