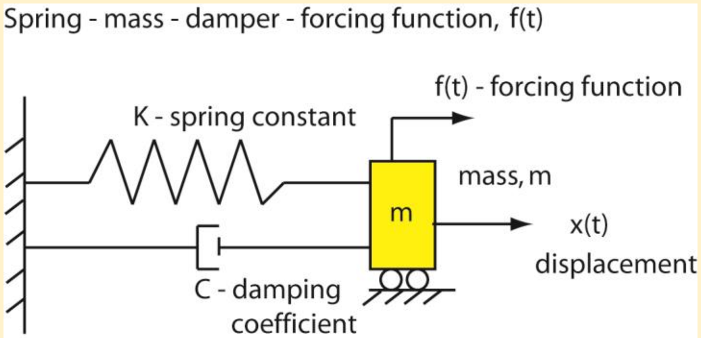
Mechanical Vibrations

In a Nut Shell: Vibrations of a spring, mass, and damper mechanical system with an applied forcing function (**forced vibration**) is an important application involving second order, linear, ordinary differential equations with constant coefficients. If there is no forcing function, f(t), then the mechanical system is said to undergo a **free vibration**.

Let x(t) be the displacement of the mechanical system (figure below) with mass m spring constant K (lb/ft, N/m), and with damping constant C (lb sec/ft or N sec/m).



For free, undamped vibration: $m\, d^2x/dt^2 + k\, x = 0$

For free, damped vibration: $m\, d^2x/dt^2 + C\, dx/dt + k\, x = 0$

For forced, undamped vibration: $m\, d^2x/dt^2 + k\, x = f(t)$

For forced, damped vibration: $m\, d^2x/dt^2 + C\, dx/dt + k\, x = f(t)$

where m = mass of the system (slugs or kg)
k = spring constant (lb/ft or N/m)
C = damping constant (lb sec /ft or N sec / m)
d^2x/dt^2 represents physically the acceleration of the mass (magnitude)
dx/dt represents physically the velocity of the mass (magnitude)
x(t) = displacement of the mass (ft or m)
f(t) = applied forcing function (lb or N)

Natural frequency: $\omega = \sqrt{(K/m)}$ rad/sec

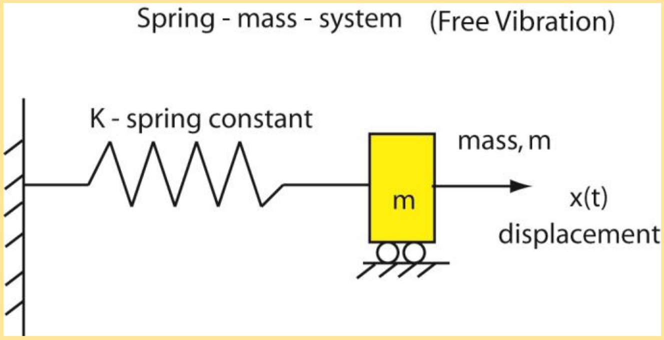
Critical damping: $C_c = 2\sqrt{(Km)}$ (lb sec/ft or N sec/m)

$C < C_c$ underdamped $C > C_c$ overdamped

Two initial conditions are needed to find the two constants of integration.

$x(0) = x_o$ and $dx(0)/dt = v_o$

Undamped, free vibration



The differential equation for undamped, free vibrations is:

$$d^2x/dt^2 + \omega_o^2 \; x = 0 \qquad \text{where} \quad \omega_o^2 = K/m$$

with initial conditions $x(0) = x_o$ and $dx(0)/dt = v_o$

Response for free vibrations depends on the spring constant, the mass, and on the initial conditions. i.e. For a solution, as usual, assume $x(t) = e^{rt}$.

The solution can be written as: (once you have found the roots for r)

$x(t) = A \cos \omega_o t + B \sin \omega_o t$ and $\omega_o = \sqrt{(k/m)}$ = natural frequency = rad/sec

where the constants of integration A and B are determined from the initial conditions.

Or the solution can be written as: $x(t) = D \cos(\omega_o t - \alpha)$

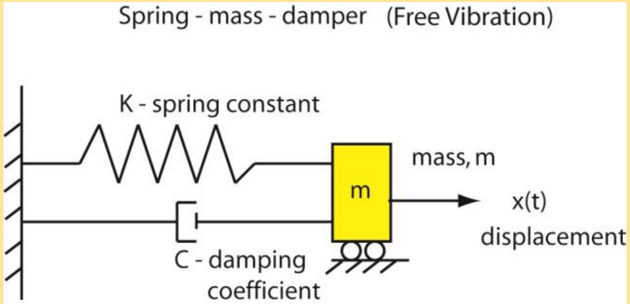
where $D = \sqrt{(A^2 + B^2)}$ $\omega_o = \sqrt{(k/m)}$ = circular frequency = rad/sec

and α = the phase angle in radians

The frequency of vibration, $v = \omega_o / 2\pi$ (cycles/sec or hertz) .

The period, T, is the time to complete once cycle . $T = 2\pi / \omega_o$ seconds.

Many mechanical systems experience friction which leads to damped vibrations of the system. The amount of damping is a major factor on the response of damped mechanical systems.



The differential equation for free vibrations with damping is:

$$d^2x/dt^2 + (C/m)\, dx/dt + \omega_o^2\ x = 0$$

with initial conditions $x(0) = x_o$ and $dx(0)/dt = v_o$

As usual to obtain a solution assume an exponential of the form, $x(t) = e^{rt}$.

The amount of **damping is a major factor** in the response of damped free vibrations of a mechanical system. There are three possible conditions for damping including underdamped, critically damped, and over-damped.

Critical damping occurs when $C^2 = 4\,k\,m$ or $C = 2\sqrt{km}$.

The solution for the overdamped case can be written as:

$x(t) = A \exp(r_1 t) + B \exp(r_2 t)$ Here exp refers to the exponential

The solution for the critically damped case can be written as:

$x(t) = \exp(-(C/2m)\,[A_1 + A_2 t]$

The solution for the underdamped case, which is of most interest, can be written as:

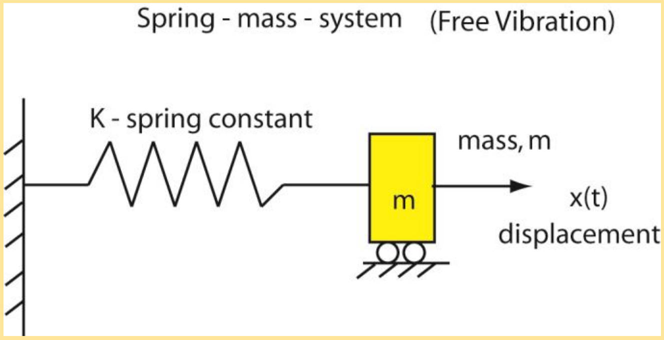
$x(t) = D\,[\exp\{(-C/2M)\}t]\,[(A/D) \cos \omega_1 t + (B/d) \sin \omega_1 t]$

where the constants of integration A and B are determined from the initial conditions.

where $D = \sqrt{(A^2 + B^2)}$ $\quad \omega_1 = [\sqrt{(4km - C^2)}] / 2m$

Here the amplitude of vibration is $D \exp(-C/2M)$ and dies out with time. So you have an oscillating response with decreasing amplitude.

Consider a spring mass system with mass, m = 1 kg. When the mass is stretched 9 cm, the force generated in the spring is 100 N. The mass is released from rest in the stretched position. Find the amplitude and frequency of the resulting motion.



The differential equation for free, undamped vibrations is:

$$d^2x/dt^2 + \omega_o^2\ x = 0$$

with initial conditions $x(0) = 9/100$ m and $dx(0)/dt = 0$

Assume $x(t) = e^{rt}$ and put into the d.e. which yields $r^2 + \omega_o^2 = 0$, and $r = \pm i\,\omega_o$

So $x(t) = A \cos \omega_o t + B \sin \omega_o t$

and $\omega_o = \sqrt{(k/m)}$ = natural frequency in rad/sec

Evaluate A and B using the initial conditions.

$$dx/dt = -A\,\omega_o \sin \omega_o t + B\,\omega_o \cos \omega_o t$$

$dx/dt\,|\,x = 0 = B\,\omega_o$ Therefore $B = 0$.

and $x(0) = A = 9/100$ m (result for the amplitude of vibration)

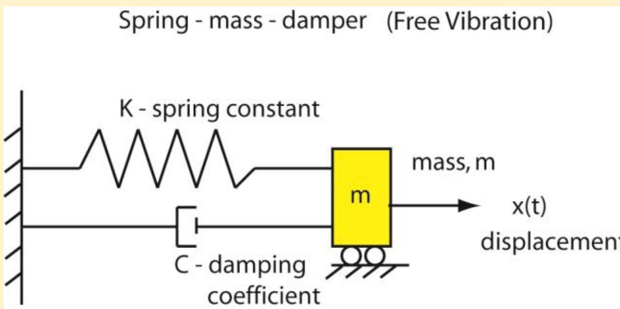
$$x(t) = (9/100) \cos \omega_o t$$

The frequency of vibration, $\nu = \omega_o / 2\pi$ (cycles/sec or hertz) .

$\nu = \sqrt{(K/m)} / 2\pi = 100/6\pi$ cycles/second (result for frequency)

Example of Free Vibration with Damping

Consider a spring mass damper system with mass, m = 1 kg and damping constant C = 4 Nsec/m. When the mass is stretched 9 cm, the force generated in the spring is 100 N. The mass is released from rest in the stretched position. Find the amplitude and frequency of the resulting motion



The differential equation is: $d^2x/dt^2 + (C/m)\, dx/dt + (k/m)\, x = 0$

with initial conditions $x(0) = 9/100$ m and $dx(0)/dt = 0$ m/sec .

Strategy: The type of vibration depends on the amount of damping. First compute the spring constant, k, and next the critical damping using $C_c = 2\sqrt{km}$. Then assume an exponential solution $x = e^{rt}$ and solve the characteristic equation for the roots of r.

F = k(extension) So $k = 100 / (9/100) = (100^2)/9$ N / m

So $C_c = 2\,(100/3) = 200 / 3$ Nsec/m Therefore $C < C_c$

and the system is underdamped. (result) $C/C_c = C / 2\sqrt{km} = 4/([2\sqrt{(100^2)/9}] = 0.06$

Next assume an exponential solution e^{rt} and put into the differential equation.

This substitution yields the characteristic equation. $r^2 + (C/m)r + k/m = 0$

with roots $r = - C/2m \pm \sqrt{[\,(C/2m)^2 - k/m\,]}$

To facilitate the solution multiply and divide by C_c .

So $r = - C/2m \pm \sqrt{[\,(C/C_c\,)^2\, 4km/4m^2 - k/m\,]} = - C/2m \pm \sqrt{\{\,[\,(C/C_c\,)^2 - 1]\; k/m\,\}}$

$r = - C/2m \pm \sqrt{- [\,1 - (C/C_c\,)^2]\, k/m} = - C/2m \pm i\{\,\sqrt{[\,1 - C^2 / 4km]\; k/m}\}$

Now $r = -C/2m \pm i\sqrt{[1 - C^2/4km]\ k/m} = -C/2m \pm i\sqrt{[1 - C^2/4km]\ \omega_o^2}$

So $r = -C/2m \pm i\ \omega_1$ where $\omega_1 = \sqrt{[1 - C^2/4km]}\ \omega_o$ and $\omega_o^2 = k/m$

The roots for r are: $r_1 = -C/2m + i\ \omega_1$ and $r_2 = -C/2m - i\ \omega_1$

Recall that the general solution is $x(t) = A \exp(r_1 t) + B \exp(r_2 t)$

where A and B are the constants of integration.

So the general solution is:

$$x(t) = \exp[(-C/2m)t]\ [A \cos \omega_1 t + B \sin \omega_1 t]$$

Note the "undamped" natural frequency $\omega_o = 100/3$ rad/sec

The "damped" natural frequency, $\omega_1 = \sqrt{[1 - C^2/4km]}\ \omega_o$

$\omega_1 = \{\sqrt{[1 - 36/10000]}\}\ (100/3)$ rad/sec Note: ω_1 slightly less than ω_o

The constants of integration, A and B, are determined from the initial conditions.

For this example $x(0) = 9/100$ m and $dx(0)/dt = 0$.

$dx/dt = -(C/2m) \exp[(-C/2m)t]\ [A \cos \omega_1 t + B \sin \omega_1 t] +$

$\exp[(-C/2m)t]\ [-\omega_1 A \sin \omega_1 t + \omega_1 B \cos \omega_1 t]$

$dx(0)/dt = 0 = -(C/2m) A + \omega_1 B$ So $B = (C/2m) A / \omega_1$.

and $x(0) = 9/100 = A$ So $B = [(4/2)(9/100)] / \{\sqrt{[1 - 36/10000]}\}(100/3)$

$B = 54 / [\sqrt{1 - 36/10000}]\ 10000$

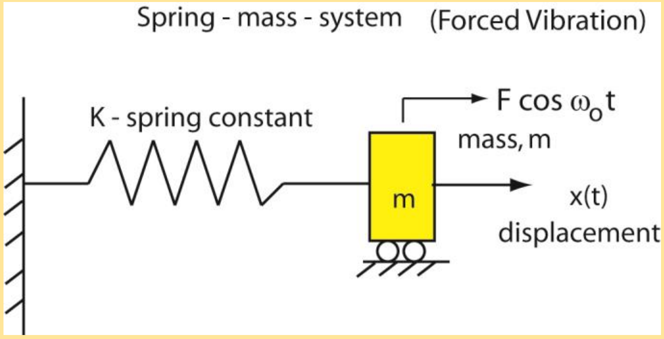
Now $x(t) = D \exp[(-C/2m)t]\ [(A/D) \cos \omega_1 t + (B/D) \sin \omega_1 t]$

where $D = \sqrt{(A^2 + B^2)}$

So the maximum amplitude starts out with a value D and decays with time according to $\exp[(-C/2m)t]$.

Example of mechanical vibration involving Resonance

Consider the undamped mechanical system with $m = 1$, $C = 0$, $K = 1$ subjected to the forcing function, $10 \cos t$. Let the initial conditions be $x(0) = 0$ and $dx(0)/dt = 0$. The differential equation is then as shown below.



The differential equation of motion is: $x'' + x = 10\cos t$ (1)

a. Find the natural frequency of the system.
b. Find the particular solution of the differential equation.
c. Find the response of the system to the forcing function for the given initial conditions.

The natural frequency = $\sqrt{(K/m)} = 1$ rad/sec. (result)

Note that the natural frequency is the same as the driving frequency of the forcing function.

The complementary solution for $x'' + x = 0$ is $x_c(t) = A\cos t + B\sin t$

For linearly independent particular solutions one cannot repeat sin t and cos t.

So the particular solution becomes $x_p(t) = C\,t\cos t + D\,t\sin t$

$dx_p(t)/dt = C\cos t + D\sin t - C\,t\,\text{sint}\,t + D\,t\cos t$

$d^2x_p(t)/dt^2 = -C\sin t + D\cos t - C\,\text{sint}\,t + D\cos t - C\,t\cos t - D\,t\sin t$

$x_p(t) = C\,t\cos t + D\,t\sin t$ (add)

$10\cos t = -2C\sin t + 2D\cos t$

Therefore $C = 0$ and $D = 5$ and thus $x_p(t) = 5\,t\sin t$ (result for b)

The response of the mechanical system is the sum of the complementary solution plus the particular solution subjected to the initial conditions.

$x(t) = x_c(t) + x_p(t)$, $x(t) = A\cos t + B\sin t + 5\,t\sin t$
$x(0) = 0 = A$, $dx(0)/dt = 0 = B$

$dx/dt = B\cos t + 5\sin t + 5\,t\cos t$

So the response of the mechanical system to the forcing function is:

$x(t) = 5\,t\sin t$ (result)

Note: The amplitude of response increases with time.

Conclusions:

1. Resonance occurs if the driving frequency of the forcing function matches that of the system's natural frequency.

2. The amplitude of motion for an undamped mechanical system grows with time when the frequency of the forcing function matches that of the natural frequency of the system.

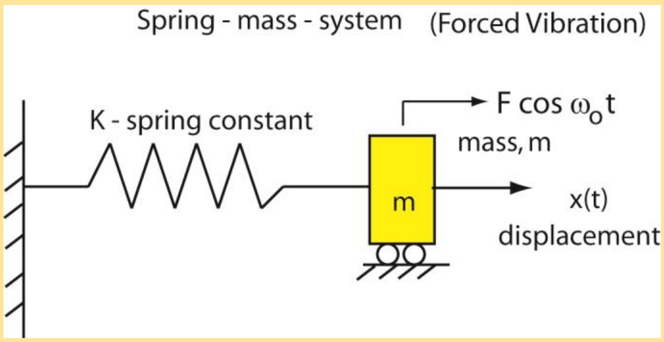
3. Start up of a mechanical system may result in a driving frequency that passes through one of the natural frequencies of the mechanical system.

4. Not shown in this example is that damping limits the build-up of amplitude of response in the forced motion of a mechanical system. In this case "practical resonance" results.

Forced Vibration, no damping, case involving "Beats"

In a Nut Shell: Under the condition where the frequency, ω_o, of the forcing function, $F \cos \omega_o t$, is close to the natural frequency of the mechanical system, $\omega = \sqrt{(K/m)}$, then a response in the nature of "beats" occurs. i.e. If two horns are not exactly tuned the same, then one hears a "beat", an audible variation in the amplitude of the combined sound. In an electrical system the variation of amplitude with time is termed, amplitude modulation.

Let x(t) be the displacement of the mechanical system (figure below) with mass m spring constant K (lb/ft, N/m), and with damping constant C (lb sec/ft or N sec/m).



For forced, undamped vibration: $m\, d^2x/dt^2 + k\,x = f(t)$

where m = mass of the system (slugs or kg)
k = spring constant (lb/ft or N/m)
d^2x/dt^2 represents physically the acceleration of the mass (magnitude)
dx/dt represents physically the velocity of the mass (magnitude)
x(t) = displacement of the mass (ft or m)
f(t) = applied forcing function (lb or N)
F = amplitude of forcing function (lb or N)
ω_o = frequency of the forcing function (rad/sec)

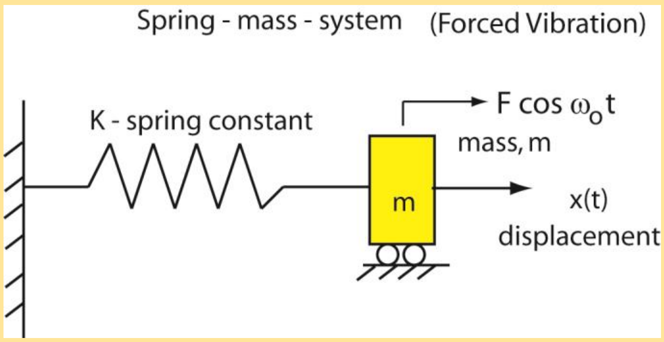
Natural frequency of mechanical system: $\omega = \sqrt{(K/m)}$ rad/sec

Two initial conditions are needed to find the two constants of integration.

$x(0) = x_o$ and $dx(0)/dt = v_o$

Example: The mass - spring system shown below has a mass, m = 1kg, and a spring rate, K = 1 N/m. The forcing function is 0.42 cos (1.1t) N. The initial conditions are: x(0) = 0 and dx(0) /dt = 0.

Let x(t) be the displacement of the mechanical system (figure below) with mass m spring constant K (lb/ft, N/m), and with damping constant C (lb sec/ft or N sec/m).



The d.e. of motion is: $d^2x/dt^2 + x = 0.42 \cos(1.1t)$

Find: a. The natural frequency of the mechanical system.
b. The particular solution, $x_P(t)$.
c. The response of the mechanical system, x(t).
d. Show a plot of the response, x(t).

Natural frequency of mechanical system: $\omega = \sqrt{(K/m)} = 1$ rad/sec

Note that the natural frequency is close to the driving frequency of 1.1 rad/sec.

The characteristic equation for the homogeneous d.e. is $r^2 + 1 = 0$

So $x_c(t) = A \cos t + B \sin t$

For the particular solution assume $x_P(t) = C \cos(1.1t) + D \sin(1.1t)$

So $d^2xP(t)/dt^2 = -1.21\, C \cos 1.1t - 1.21\, D \sin 1.1t$

and $d^2x_P(t)/dt^2 + x_P(t) = -0.21\, C \cos(1.1t) - 0.21\, D \sin(1.1t) = 0.42 \cos(1.1t)$

Thus $C = -2$ and $D = 0$

The result for the particular solution is $x_P(t) = -2 \cos(1.1t)$

The response is: $x(t) = x_c(t) + x_P(t)$

$x(t) = A \cos t + B \sin t - 2 \cos 1.1t$ and $dx(t)/dt = - A \sin t + B \cos t - 2.2 \sin 1.1t$

For initial conditions: $x(0) = 0 = A - 2$, So $A = 2$, $dx(0)/dt = 0$, $B = 0$

Thus the response of the mechanical system, $x(t)$, is:

$$x(t) = 2 \cos (t) - 2 \cos (1.1t) \qquad \text{(result)}$$

This result can also be expressed as a product of two sine functions using the trig identies:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

and $$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

where $a + b = (1.1t)$ and $a - b = (1t)$

So $a = 1.05t$ and $b = 0.05t$

And $\cos (1.1t) = \cos (1.05t) \cos (0.05t) - \sin (1.05t) \sin (0.05t)$

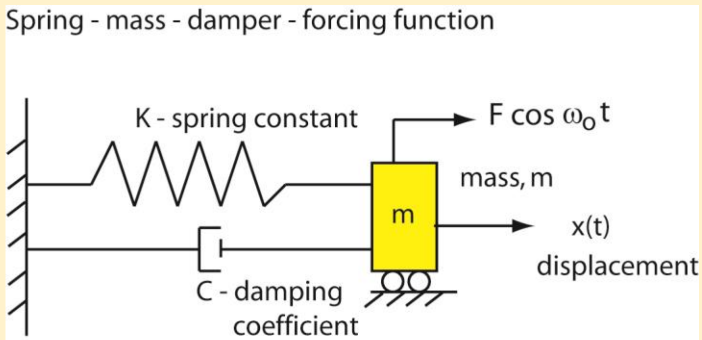
$\cos(1t) = \cos (1.05t) \cos (0.05t) + \sin (1.05t) \sin (0.05t)$

So $x(t) = 2 [\cos (t) - \cos (1.1t)] = 4[\sin (1.05t) \sin(0.05t)]$

The "envelope" is governed by sin(0.05t) with a period of 40π seconds and shows the beats (or modulation). The "High Frequency"vibration within the "envelope" is governed by sin (1.05t) with a period of 1.9π seconds.

In a Nut Shell: Vibrations of a spring, mass, and damper mechanical system with an applied forcing function (**forced vibration**) is an important application involving second order, linear, ordinary differential equations with constant coefficients. The forcing function generally controls the response of the mechanical system since the free vibration with damping will diminish with time. Practical resonance occurs when the frequency of the forcing function is near that of the undamped natural frequency, $\sqrt{(K/m)}$.

Let x(t) be the displacement of the mechanical system (figure below) with mass m spring constant K (lb/ft, N/m), and with damping constant C (lb sec/ft or N sec/m).



For forced, undamped vibration: $m\, d^2x/dt^2 + k\,x = f(t)$

For forced, damped vibration: $m\, d^2x/dt^2 + C\, dx/dt + k\,x = f(t)$

where m = mass of the system (slugs or kg)
k = spring constant (lb/ft or N/m)
C = damping constant (lb sec /ft or N sec / m)
d^2x/dt^2 represents physically the acceleration of the mass (magnitude)
dx/dt represents physically the velocity of the mass (magnitude)
x(t) = displacement of the mass (ft or m)
f(t) = applied forcing function (lb or N)
F = amplitude of forcing function (lb or N)
ω_o = frequency of the forcing function (rad/sec)

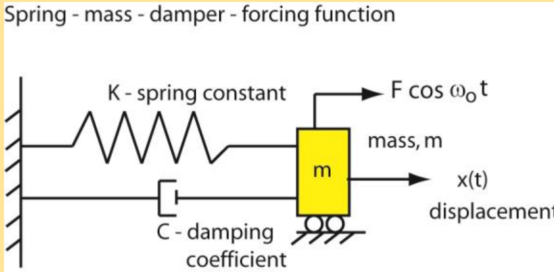
Natural frequency of mechanical system: $\omega = \sqrt{(K/m)}$ rad/sec

Two initial conditions are needed to find the two constants of integration.

$x(0) = x_o$ and $dx(0)/dt = v_o$

Example: Mechanical System with Damping - - Practical Resonance

Consider the damped mechanical system with m = 1, C = 4, K = 8 subjected to the forcing function, $8\sqrt{8}\cos\sqrt{8}\,t$. Let the initial conditions be x(0) = 0 and dx(0)/dt = 0. The differential equation is then as shown below.



The differential equation of motion is: $x'' + 4x' + 8x = 8\sqrt{8} \cos \sqrt{8}\, t$

a. Find the natural frequency of the system.
b. Find the particular solution, $x_p(t)$ of the differential equation.
c. Find the response of the system to the forcing function for the given initial conditions.

The natural frequency = $\sqrt{(K/m)} = \sqrt{8}$ rad/sec. (result)

Note that the natural frequency is the same as the driving frequency of the forcing function.

The complementary solution for $x'' + 4x' + 8x = 0$ from the characteristic equation

$r^2 + 4r + 8 = 0$ with roots $r = -2 \pm 4\, i$ is:

$x_c(t) = \exp(-2t)\,[A \cos 4t + B \sin 4t]$ Note: This transient response diminishes with time

leaving only the particular solution as the steady state response after a short period of time.

So the particular solution becomes $x_p(t) = C \cos \sqrt{8}\, t + D \sin \sqrt{8}\, t$

$dx_p(t)/dt = -\sqrt{8} C \sin \sqrt{8} t + \sqrt{8}\, D \cos \sqrt{8} t$

$d^2x_p(t)/dt^2 = -8C \cos \sqrt{8} t - 8D \sin \sqrt{8} t$

$4\, dx_p(t)/dt = 4\sqrt{8}\, D \cos \sqrt{8} t - 4\sqrt{8} C \sin \sqrt{8} t$

$8\, x_p(t) = 8\, C \cos \sqrt{8}\, t + 8\, D \sin \sqrt{8}\, t$

$8\sqrt{8} \cos \sqrt{8}\, t = 4\sqrt{8}\, D \cos \sqrt{8} t - 4\sqrt{8} C \sin \sqrt{8} t$ So $C = 0$ and $D = 2$

Result: $x_p(t) = 2 \sin \sqrt{8}\, t$ Steady state response after free vibration damps out.