Existence and Uniqueness of Solutions / Exact Differential Equations

In a Nut Shell: Not all first order differential equations have a solution nor a unique solution. The existence and uniqueness theorem provides the conditions for which the differential equation has a solution and whether it is unique.

Theorem for a general first order, ordinary differential equation of the form:

$$dy/dx = f(y,x) \qquad y(a) = b$$

This differential equation has a solution as long as $f(y,x)$ is continuous in a neighborhood of $y = b$, $x = a$. This solution is unique provided $\partial f/\partial y$ is also continuous in the same neighborhood.

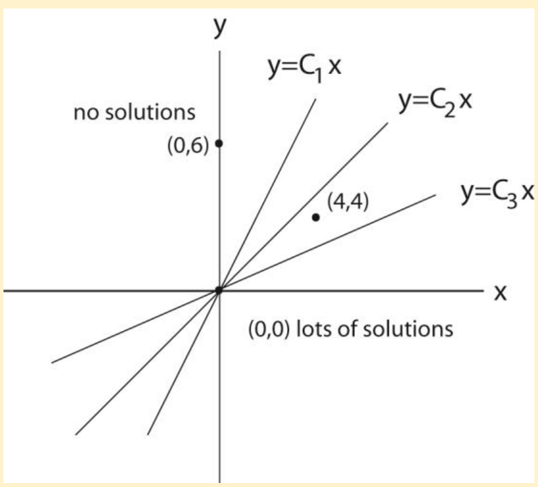
Note: The d.e. might be linear or nonlinear depending on $f(y,x)$.

Example: Determine the existence and uniqueness of the solution for the following d.e.

$$x\, dy/dx = y \qquad y(a) = b$$

Rewrite the d.e. as $dy/dx = y/x = f(x,y)$

So $f(x,y = y/x$ is continuous near (a,b) provided $a \neq 0$.



Results in figure above: Solution exists and is unique at (4,4). No solutions at (0,6). Infinitely many solutions passing through (0,0)

Example: Determine the existence and uniqueness of the solution for the following d.e.

$$x\, dy/dx = 2\, y \qquad y(a) = b$$

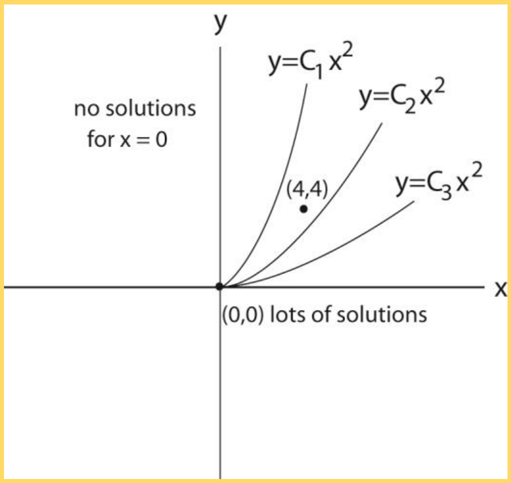
Rewrite the d.e. as $dy/dx = 2y/x = f(x,y)$

So $f(x,y) = 2y/x$ is continuous near (a,b) provided $a \neq 0$.

Therefore a solution, $y(x)$, exists provided $a \neq 0$.

$\partial f/\partial y = 2/x$ which is also continuous provided $a \neq 0$. So a solution exists that

is unique provided $a \neq 0$.



Results in figure above: Solution exists and is unique at (4,4). No solutions on $x = 0$. Infinitely many solutions passing through (0,0) (parabolas)

Note: The existence and uniqueness theorem gives required conditions that a solution exists and is unique. **It fails to show that a solution does not exist.** In fact it does in this example at (0,0).

Solution of Exact Differential Equations

In a Nut Shell: A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is exact if $\partial M/\partial y = \partial N/\partial x$. The strategy to solve an exact differential equation is to seek an implicit solution in the form of a level curve. i.e. $F(x,y) = C$. Envision a function (surface) described by $F(x,y)$. Then $F(x,y) = C$ is a level curve on the surface. An example of a level curve is an isocline on the surface of a hill.

Procedure: Recall the definition of a differential $dF = (\partial F/\partial x)\, dx + (\partial F/\partial y)\, dy$.
Set $M = (\partial F/\partial x)$ and $N = (\partial F/\partial y)$ and seek a solution for $F(x,y)$ given an initial condition $y(x_0) = y_0$. Note i.e. integration of $(\partial F/\partial x)$ will yield a "constant of integration" $g(y)$. Then calculate $\partial F/\partial y$ and set it equal to the value of $N(x,y)$ to find the function $g(y)$. Consider the example given below.

Example: Determine if the following differential equation is exact and if so, find its solution.

$$dy/dx = -(1 + ye^{xy}) / (2y + x e^{xy}) \quad \text{with } y(0) = 1$$

Rewrite as $(1 + ye^{xy})\, dx + (2y + x e^{xy})\, dy = 0$ with the initial condition $y(0) = 1$

So $M = (1 + ye^{xy})$ and $N = (2y + x e^{xy})$

$\partial M/\partial y = e^{xy} + xy\, e^{xy}$ and $\partial N/\partial x = e^{xy} + xy\, e^{xy}$ So the differential equation is exact.

Now use the total differential $dF = (\partial F/\partial x)\, dx + (\partial F/\partial y)\, dy = 0$ since $F(x,y) = C$

where $\partial F/\partial x = M = (1 + ye^{xy})$ and $\partial F/\partial y = N = (2y + x e^{xy})$

$\partial F/\partial x = 1 + ye^{xy}$ and $\partial F/\partial y = 2y + x e^{xy}$

Integrate the expression for $\partial F/\partial x$ with respect to x keeping in mind that y will be a constant.

$F(x,y) = x + e^{xy} + g(y)$ Next take the partial derivative with respect to y

$\partial F/\partial y = x e^{xy} + dg/dy$ and set it equal to $\partial F/\partial y = 2y + x e^{xy}$

So $dg/dy = 2y$ and therefore $g(y) = y^2$

The result is $F(x,y) = x + e^{xy} + y^2 = C$

Apply the initial condition $y(0) = 1$ Therefore $C = 1$

The solution of the exact differential equation is $x + e^{xy} + y^2 = 1$ (result)

Note: One could also start by integrating the expression for $\partial F/\partial y$ and follow a similar procedure giving the same final result.