Basics of Triple Integrals

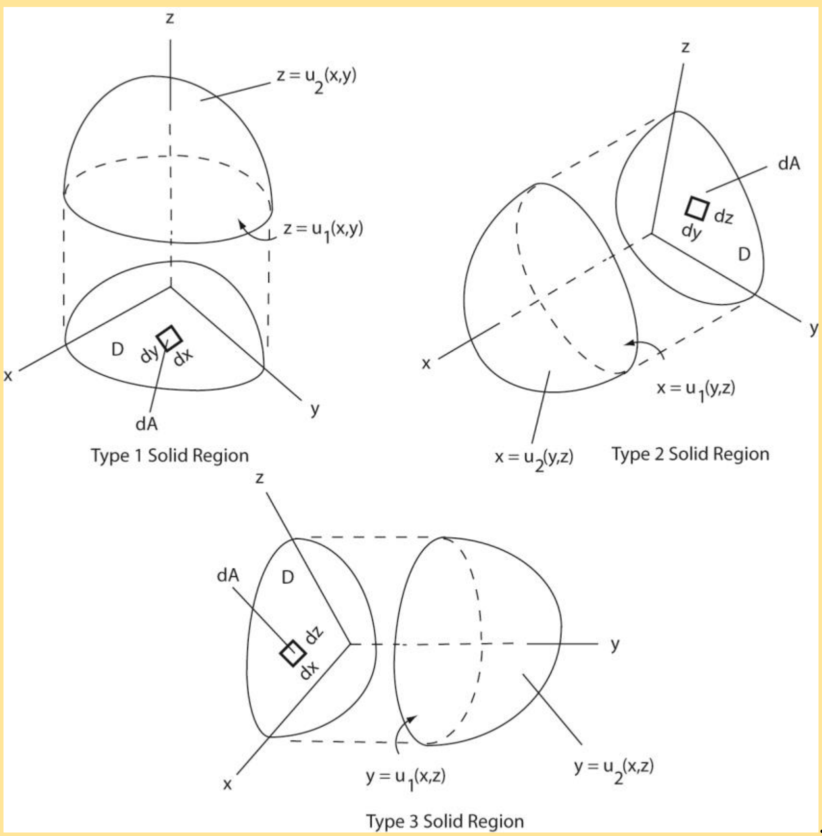
In a Nut Shell: The triple integral of a function, f(x,y,z), gives the value of the function integrated over the region of the volume. The element of volume, dV, in rectangular cartesian coordinates can be expressed as dV = dx dy dz or in any of six possible combinations - dx dy dz, dy dx dz, dy dx dz, dy dz dx, dz dx dy, and dz dy dx .

Cylindrical and spherical coordinates may also be used during integration and may be more useful depending upon the application.

Strategy: The first step is to identify the **type of region** over which the function is to be integrated. This important step determines the first variable for integration.

There are three types of regions called **Type 1, Type 2, and Type 3** as shown below.

- A **Type 1** solid is one where its projection is on the xy-plane.
  The first integration is in the z-direction. $I = \int \int \int f(x,y,z)\ dz\ dA$
- A **Type 2** solid is one where its projection is on the yz-plane
  The first integration is in the x-direction. $I = \int \int \int f(x,y,z)\ dx\ dA$
- A **Type 3** solid is one where its projection is on the xz-plane.
  The first integration is in the y-direction. $I = \int \int \int f(x,y,z)\ dy\ dA$



In a Nut Shell: No matter the type of region of integration each triple integral contains **three distinct parts.** They include the inner integral, the middle integral, and the outer integral. The table below illustrates these parts for a Type 1 region.

The **"inner integral"** of $\int \int [\int f(x,y,z)\, dz\ ]\ dA$ is the one in the brackets.

When evaluating this part of the triple integral the independent variables, x and y (within dA), are held constant as if the function f(x,y,z) only depended on z. The limits of integration for the inner integral are from $z = u_1(x,y)$ to $z = u_2(x,y)$.

Inner integral $$\int_{z = u_1(x,y)}^{z = u_2(x,y)} f(x,y,z)\, dz$$

The **"middle integral"** of $\int [\int \int f(x,y,z)\, dz\, dx\ ]\ dy$ is the one in the brackets.
When evaluating this portion of the triple integral the independent variable y is held constant as if the function f(x,y,z) only depended on x.

Middle integral $$\int_{x = v_1(y)}^{x = v_2(y)} [\int_{z = u_1(x,y)}^{z= u_2(x,y)} f(x,y,z)\, dz\ ]\ dx\ \ dy$$

Finally the **"outer integral"** is $\int\int\int f(x,y,z)\, dz\, dx\, dy$. When evaluating this final portion of the integral the last variable, in this case, y, must go from one constant to another. i.e.

Outer integral $$\int_{y = a}^{y = b} \int_{x = v_1(y)}^{x = v_2(y)} [\int_{z = u_1(x,y)}^{z = u_2(x,y)} f(x,y,z)\, dz\ ]\ dx\ \ dy$$

One of the challenges is determining the limits of integration.

One strategy is to project the volume onto a plane, determine the intersections on the projected plane, and use these expressions to establish the limits of integration.

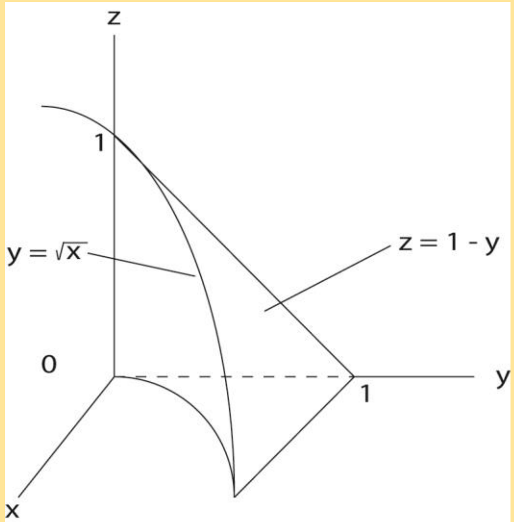
Example: Determine the limits of integration using a **Type 1 Solid region** for the triple integral

$$I = \int \int \int [f\, dz]\ dA$$ where the region (shown below) is

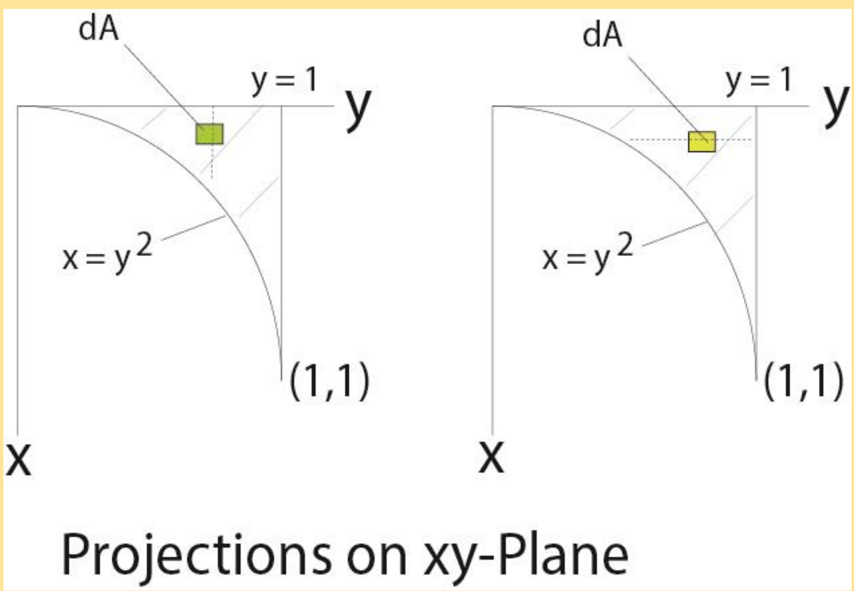
bounded by the plane $z = 1 - y$, the surface $y = \sqrt{x}$, the yz-plane and the xy-plane.

The first integration for Type 1 Solid regions is in the z-direction. In this example,

start by "sweeping" the element of volume, dz dA from $z = 0$ to $z = 1 - y$.



To determine the limits of integration in the x and y directions, project the solid region on to the xy-plane and examine the intersections. The order of integration for x and y is optional. See the figures below.

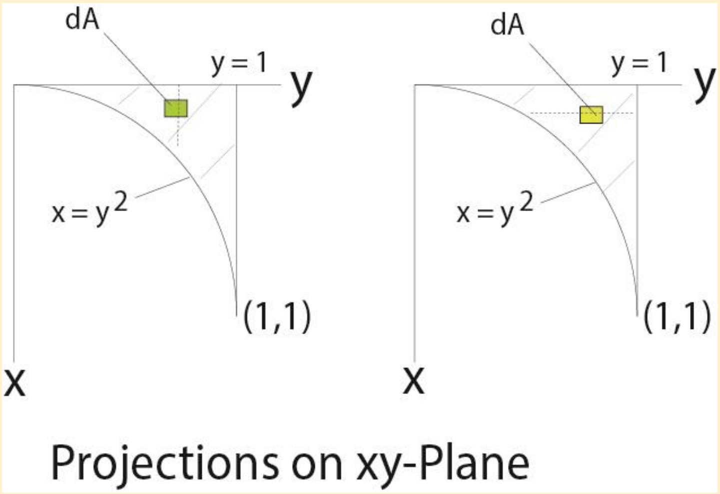


The first integration (for Type 1 Solid Region) is in the z-direction.

$$I = \int\int \int_{z=0}^{z=1-y} [\, f\, dz \,]\; dA$$

Now the element of area, dA, may be either dx dy or dy dx (depending on the order of integration). So determine the limits of integration in the x and y – directions by projecting the solid region on to the xy-plane as shown below.

Note: The limits of integration on the inner integral contain at most two variables, the limits of integration on the middle integral contain at most one variable, and the limits of integration on the outer integral must be constants.



Option 1: By inspection of the intersections, the limits of integration in the x-direction (figure on the left) are $x = 0$ to $x = y^2$. Next "sweep" the element of area in the y-direction.

In this case the limits of integration are from $y = 0$ to $y = 1$. So the integral becomes

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=y^2} \int_{z=0}^{z=1-y} [\, f(x,y,z)\, dz \,]\; dx\; dy \qquad \text{(result)}$$

Option 2: By inspection of the intersections, the limits of integration in the y-direction (figure on the right) give $y = \sqrt{x}$ to $y = 1$. Then "sweep" the element of area in the x-direction.

In this case the limits of integration are from $x = 0$ to $x = 1$. So the integral becomes

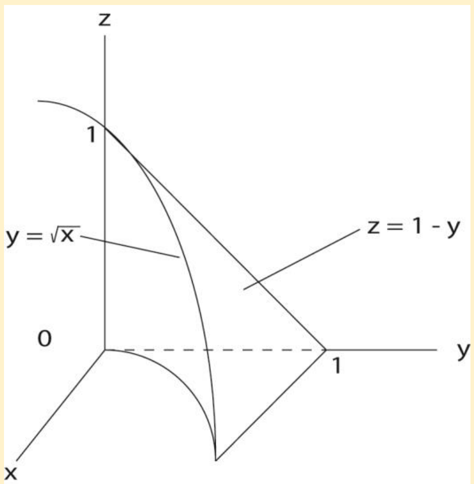
$$I = \int_{x=0}^{x=1} \int_{y=\sqrt{x}}^{y=1} \int_{z=0}^{z=1-y} [\, f(x,y,z)\, dz \,]\; dy\; dx \qquad \text{(result)}$$

Example: Determine the limits of integration using a **Type 2 Solid region** for the triple integral

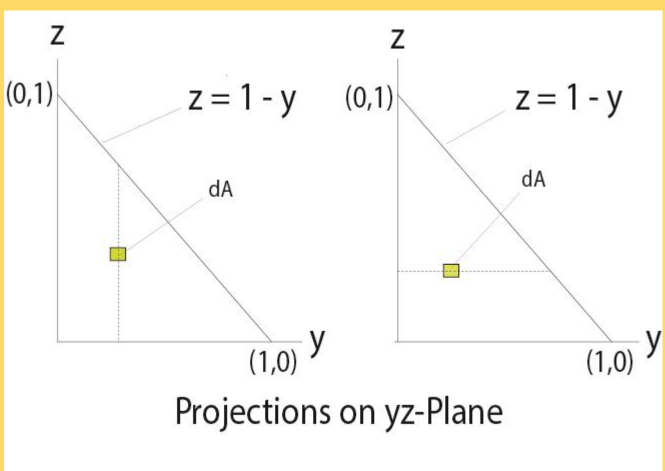
$$I = \int\int\int [\, f\, dx\,]\; dA$$ where the region (shown below) is

bounded by the plane $z = 1 - y$, the surface $y = \sqrt{x}$, the yz-plane and the xy-plane.

The first integration for Type 2 Solid regions is in the x-direction. In this example, start by "sweeping" the element of volume, dx dA from $x = 0$ to $x = y^2$.



To determine the limits of integration in the y and z directions, project the solid region on to the yz-plane and examine the intersections. The order of integration for y and z is optional. See the figures below.

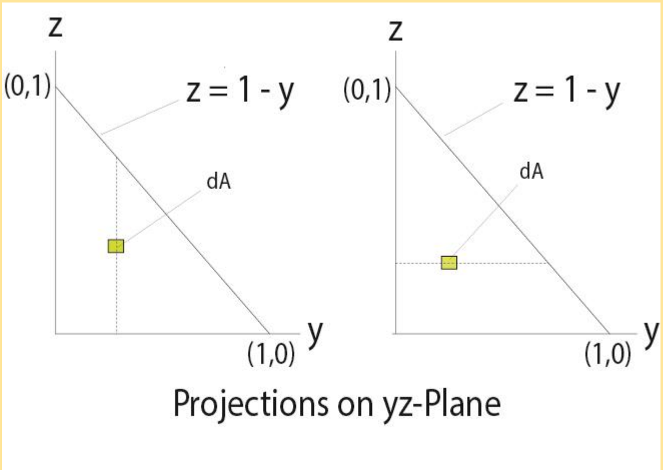


The first integration (for Type 2 Solid Region) is in the z-direction.

$$I = \iint \int_{x=0}^{x=y^2} [\, f\, dx \,] \; dA$$

Now the element of area, dA, may be either dy dz or dz dy (depending on the order of integration). So determine the limits of integration in the y and z – directions by projecting the solid region on to the yz-plane as shown below.

Note: The limits of integration on the inner integral contain at most two variables, the limits of integration on the middle integral contain at most one variable, and the limits of integration on the outer integral must be constants.



Option 1: By inspection of the intersections, the limits of integration in the z-direction (figure on the left) are z = 0 to z = 1 - y . Next "sweep" the element of area in the y-direction. In this case the limits of integration are from y = 0 to y = 1. So the integral becomes

$$I = \int_{y=0}^{y=1} \int_{z=0}^{z=1-y} \int_{x=0}^{x=y^2} [\, f(x,y,z)\, dx \,] \; dz \; dy \qquad \text{(result)}$$

Option 2: By inspection of the intersections, the limits of integration in the y-direction (figure on the right) givey = 0 to y = 1 - z . Then "sweep" the element of area in the z-direction. In this case the limits of integration are from z = 0 to z = 1. So the integral becomes

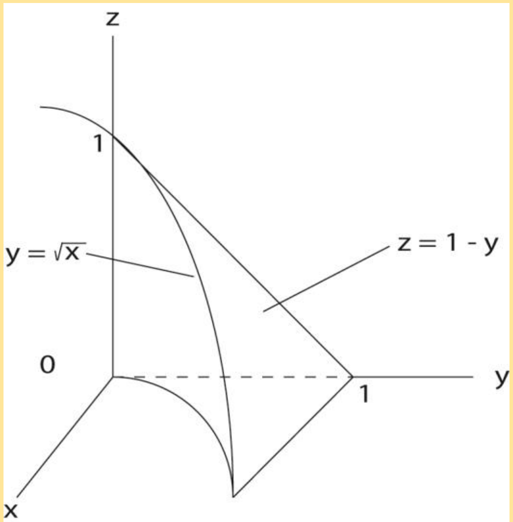
$$I = \int_{z=0}^{z=1} \int_{y=0}^{y=1-z} \int_{x=0}^{x=y^2} [\, f(x,y,z)\, dx \,] \; dy \; dz \qquad \text{(result)}$$

Example: Determine the limits of integration using a **Type 3 Solid region** for the triple integral

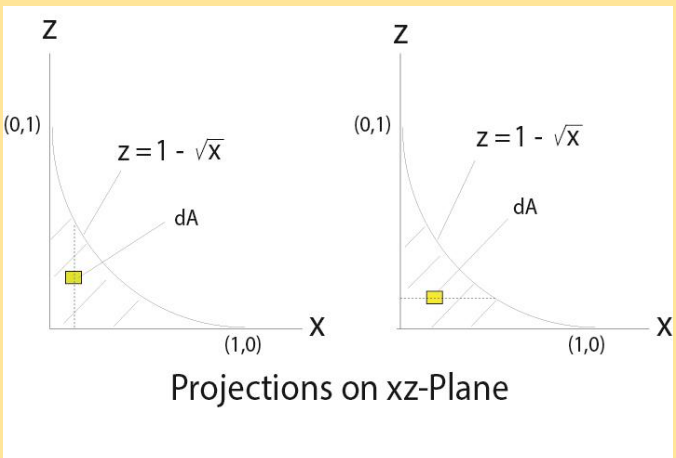
$$I = \int\int\int [\, f\, dy \,]\; dA$$ where the region (shown below) is

bounded by the plane $z = 1 - y$, the surface $y = \sqrt{x}$, the yz-plane and the xy-plane.

The first integration for Type 3 Solid regions is in the y-direction. In this example, start by "sweeping" the element of volume, dy dA from $y = \sqrt{x}$ to $y = 1 - z$.



To determine the limits of integration in the x and z directions, project the solid region on to the xz-plane and examine the intersections. The order of integration for x and z is optional. See the figures below.

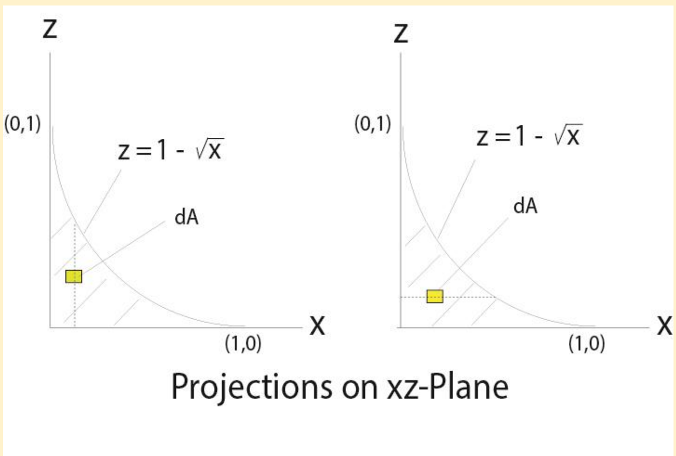


The first integration (for Type 3 Solid Region) is in the y-direction.

$$I = \int\int \int_{y=\sqrt{x}}^{y=1-z} [\, f\, dy \,] \; dA$$

Now the element of area, dA, may be either dx dz or dz dx (depending on the order of integration). So determine the limits of integration in the x and z – directions by projecting the solid region on to the xz-plane as shown below.

Note: The limits of integration on the inner integral contain at most two variables, the limits of integration on the middle integral contain at most one variable, and the limits of integration on the outer integral must be constants.



Option 1: By inspection of the intersections, the limits of integration in the z-direction (figure on the left) are $z = 0$ to $z = 1 - \sqrt{x}$. Next "sweep" the element of area in the x-direction. In this case the limits of integration are from $x = 0$ to $x = 1$. So the integral becomes

$$I = \int_{x=0}^{x=1} \int_{z=0}^{z=1-\sqrt{x}} \int_{y=\sqrt{x}}^{y=1-z} [\, f(x,y,z)\, dy \,] \; dz \; dx \qquad \text{(result)}$$

Option 2: By inspection of the intersections, the limits of integration in the x-direction (figure on the right) give $x = \sqrt{x}$ to $x = (1 - z)^2$. Then "sweep" the element of area in the z-direction. In this case the limits of integration are from $z = 0$ to $z = 1$. So the integral becomes

$$I = \int_{z=0}^{z=1} \int_{x=0}^{x=(1-z)^2} \int_{y=\sqrt{x}}^{y=1-z} [\, f(x,y,z)\, dy \,] \; dx \; dz \qquad \text{(result)}$$