Initial Value, Boundary Value, and Eigenvalue Problems

In a Nut Shell: There are three important types of problems involving linear, second order, ordinary, homogeneous d.e.'s of the form

$$y'' + p(x)y' + q(x)y = 0$$

that frequently appear in a first course in differential equations. They include:

Initial Value Problems
Boundary Value Problems
Eigenvalue Problems.

Strategy: Start by identifying the type of problem. Each type will be described followed by an example.

Type 1: An initial value problem has conditions: $y(a) = A$, $y'(a) = B$

You saw this type of problem in studying free vibrations of a mechanical system.

$my'' + c\,y' + k\,y = 0$ along with conditions:

$y(0) = y_o$, $dy(0)/dt = v_o$ (initial displacement and initial speed of system)

where: m = mass, c = damping coefficient, and k = spring rate
The strategy is to find the complementary solution subject to the initial conditions.

Example: Find the solution for the following **initial value problem**.

$$d^2y/dx^2 + 6dy/dx + 13y = 0$$

$y(0) = 0$, $y'(0) = 1$ (initial values)

You can write this d.e. in operator notation $(D^2 + 6D + 13)\,y = 0$

Assume $y = Ae^{rx}$ for the complementary solution.

So $d^2y/dx^2 = A\,r^2\,e^{rx}$, $dy/dx = A\,r\,e^{rx}$, and $y = Ae^{rx}$

Substitute into the d.e. yields

$$Ae^{rx}\,(r^2 + 6r + 13) = 0$$

Since $Ae^{rx} \neq 0$, the characteristic equation for r becomes:

$$r^2 + 6r + 13 = 0,$$

with roots $-3 \pm 2i$ using the quadratic formula.

The complementary solution, y_c, is:

$$y_c(x) = Fe^{(-3 + 2i)x} + Ge^{(-3 - 2i)x}$$

where F and G are undetermined constants (need two initial conditions)

which can be expressed as follows, (equivalent complementary solution)

$$y(x) = y_c(x) = e^{-3x} (C_1 \sin 2x + C_2 \cos 2x)$$

Apply the initial values to find C_1 and C_2. $y(0) = 0$ gives $0 = C_2$

So $y'(x) = -3 e^{-3x} (C_1 \sin 2x) + 2 e^{-3x} (C_1 \cos 2x)$

and $y'(0) = 1 = 2 C_1$ so $C_1 = 1/2$

The resulting solution for the initial value problem is $y(x) = (1/2) e^{-3x} \sin (2x)$

Type 2: A Boundary Value problem has conditions: $y(a) = A$, $y(b) = B$

Note: The boundary value problem also goes under the name of an end point problem.

Other possible boundary value conditions (or endpoint conditions) include:

$y'(a) = A$, $y(b) = B$, or $y'(a) = A$, $y'(b) = B$, or any linear combination

The procedure for solution is to find the complementary solution subject to the end conditions.

Example: Find the solution for the following **boundary value problem**.

$$d^2y/dx^2 + 6\, dy/dx + 13 y = 0$$

$$y(0) = 0, \quad y'(\pi) = 1 \qquad \text{(boundary values)}$$

You can write this d.e. in operator notation $(D^2 + 6D + 13) y = 0$

Assume $y = Ae^{rx}$ for the complementary solution.

So $d^2y/dx^2 = A r^2 e^{rx}$, $dy/dx = A r e^{rx}$, and $y = Ae^{rx}$

Substitute into the d.e. yields: $Ae^{rx} (r^2 + 6r + 13) = 0$

Since $Ae^{rx} \neq 0$, the characteristic equation for r becomes:

$$r^2 + 6r + 13 = 0,$$

with roots $-3 \pm 2i$ using the quadratic formula.

The complementary solution, y_c, is:

$$y_c(x) = Fe^{(-3+2i)x} + Ge^{(-3-2i)x}$$

where F and G are undetermined constants (need two initial conditions)

which can be expressed as follows, (equivalent complementary solution)

$$y(x) = y_c(x) = e^{-3x}(C_1 \sin 2x + C_2 \cos 2x)$$

Apply the initial values to find C_1 and C_2. $y(0) = 0$ gives $0 = C_2$

So $y'(x) = -3e^{-3x}(C_1 \sin 2x) + 2e^{-3x}(C_1 \cos 2x)$

and $y'(0) = 1 = 2C_1$ so $C_1 = 1/2$

The resulting solution for the initial value problem is $y(x) = (1/2)e^{-3x}\sin(2x)$

In a Nut Shell: An **eigenvalue problem** is special type of boundary value problem (endpoint problem) with an unknown parameter, λ, where the differential equation has the following form along with the associated boundary conditions:

Type 3: An eigenvalue problem

$$y'' + p(x)y' + \lambda q(x)y = 0$$

$$y(a) = A, \quad y(b) = B$$

where λ is a parameter, the eigenvalues, yet to be determined.

(The goal is to find values of λ that yield nontrivial solutions of the d.e.)

Example: Solve the following eigenvalue problem for eigenvalues and eigenvectors.

$$y'' + \lambda y = 0$$

$$y(0) = 0, \quad y(L) = 0 \qquad \text{here } L > 0$$

Note: There are three possibilities for the unknown eigenvalues, λ. For example

λ could be zero, negative, or positive. One must consider each case.

Case 1: $\lambda = 0$, Therefore $y = Ax + B$

$$y(0) = 0 = B, \quad y(L) = 0 = AL, \quad \text{Therefore } A = 0$$

so $y(x) = 0$ (a trivial solution). There are no eigenvalues (λ) or eigenfunctions.

Case 2: $\lambda = -\alpha^2$ The d.e. becomes $y'' - \alpha^2 y = 0$; $r^2 = \alpha^2$

So $y(x) = A \cosh \alpha x + B \sinh \alpha x$

$y(0) = 0 = A$, $y(L) = 0 = B \sinh \alpha L$

Either $B = 0$ or $\sinh \alpha L$ But $\sinh \alpha L \neq 0$, So B must be zero.

Again the solution remains as the trivial solution $y(x) = 0$ (No eigenvalues)

Case 3: $\lambda = \alpha^2$ $y'' + \alpha^2 y = 0$ so $r^2 = -\alpha^2$

$y(0) = 0$, $y(L) = 0$ here $L > 0$

$y(x) = A \sin \alpha x + B \cos \alpha x$

$y(0) = 0 = B$, and $y(L) = 0 = A \sin \alpha L$

Therefore either $A = 0$ or $\sin \alpha L = 0$

But for a nontrivial solution ($y(x) \neq 0$), $A \neq 0$

So $\sin \alpha L = 0$, which holds for: $\alpha L = n\pi$, $n = 1, 2, 3, \ldots$

Therefore the eigenvalues are: $\lambda_n = \alpha^2 = (n\pi/L)^2$

And the associated eigenfunctions are: $y_n = \sin(n\pi x/L)$

Example: Solve the following eigenvalue problem for eigenvalues and eigenvectors.

$$y'' + \lambda y = 0 \quad \text{where} \quad 0 < x < L$$

$$y(0) = 0$$

$$h\,y(L) + y'(L) = 0 \quad \text{where} \quad h > 0$$

Note: There are three possibilities for the unknown eigenvalues, λ. For example λ could be zero, negative, or positive. One must consider each case.

Case 1: $\lambda = 0$, Therefore $y = Ax + B$

$y(0) = 0 = B$, $y'(L) = A$

$h\,y(L) + y'(L) = 0 = hAL + A = A(hL + 1)$

Since $hL + 1 \neq 0$, $A = 0$ and $y(x) = 0$

Therefore there are no eigenvalues or eigenfunctions for case 1.

Case 2: $\lambda = -\alpha^2$ The d.e. becomes $y'' - \alpha^2 y = 0$; $r^2 = \alpha^2$

So $y(x) = A \cosh \alpha x + B \sinh \alpha x$

$$y(0) = 0 = A \quad \text{and} \quad y'(x) = \alpha B \cosh \alpha x$$

$$h\, y(L) + y'(L) = 0 = h B \sinh \alpha L + \alpha B \cosh \alpha L$$

$$B(h \sinh \alpha L + \alpha \cosh \alpha L) = 0$$

So $B(\tanh \alpha L + \alpha / h) = 0$

And either $B = 0$ or $\tanh \alpha L = -\alpha / h$

But $\tanh \alpha L \geq 0$ so $B = 0$ and $y(x) = 0$ (trivial solution)

Therefore there are no eigenvalues nor eigenfunctions for case 2.

Case 3: $\lambda = \alpha^2$ The d.e. becomes $y'' + \alpha^2 y = 0$; $r^2 = -\alpha^2$

So $y(x) = A \cos \alpha x + B \sin \alpha x$

With boundary conditions:

$$y(L) = B \sinh \alpha L$$

$$y(0) = 0 = A \quad \text{and} \quad y'(x) = \alpha B \cos \alpha x$$

$$h\, y(L) + y'(L) = 0 = h B \sin \alpha L + \alpha B \cos \alpha L$$

$$B(h \sin \alpha L + \alpha \cos \alpha L) = 0$$

So $B(\tan \alpha L + \alpha / h) = 0$

And either $B = 0$ or $\tan \alpha L = -\alpha / h = -\alpha L / hL$

For a nontrivial solution $B \neq 0$ so $\tan \alpha L = -\alpha L / hL$

Let $\beta = \alpha L$ so $\tan \beta_n = -\beta_n / hL$

$\alpha_n = \beta_n / L$ and $\lambda_n = \alpha_n^2 = (\beta_n / L)^2$ = eigenvalues

$y_n = \sin(\beta_n x / L)$ = eigenfunctions

Note: Eigenvalues are determined graphically by the intersection of $\tan \beta_n$ and $-\beta_n / hL$