Dirichlet Applications - Heat Conduction in a Plate

In a Nut Shell: The governing equation for heat conduction in a plate is:

$$\partial u/\partial t \quad = \quad k\,[\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2\,] \text{ ----------------------- } (1)$$

where $u = u(x,y,t)$ = the temperature in the plate at any time t
x,y = the location in the plate
t = the time at which the temperature at x is $u(x,y,t)$
and k is the thermal diffusivity of the material

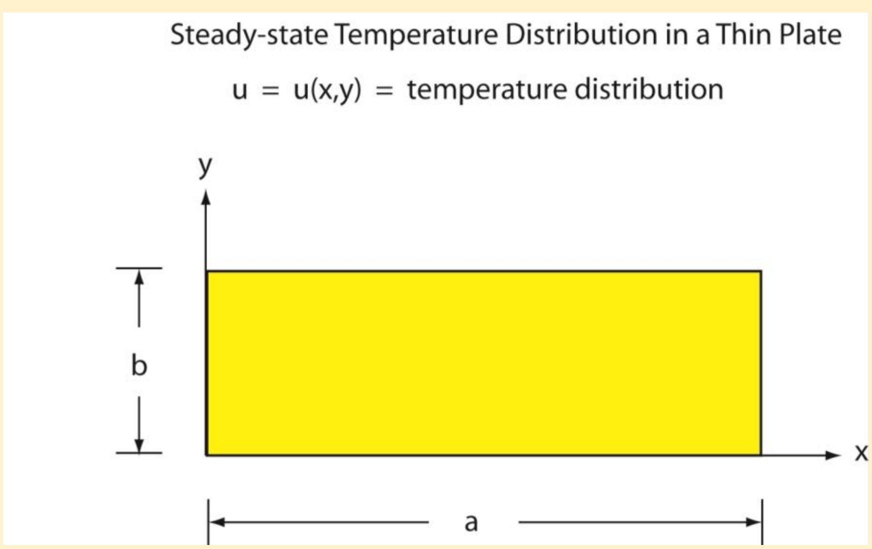
The desired outcome is to predict the temperature distribution, $u(x,y,t)$, in the plate as a function of time, t.

For steady-state heat conduction, $\partial u/\partial t = 0$. So the **steady-state temperature distribution in the plate is governed by Laplace's equation**:

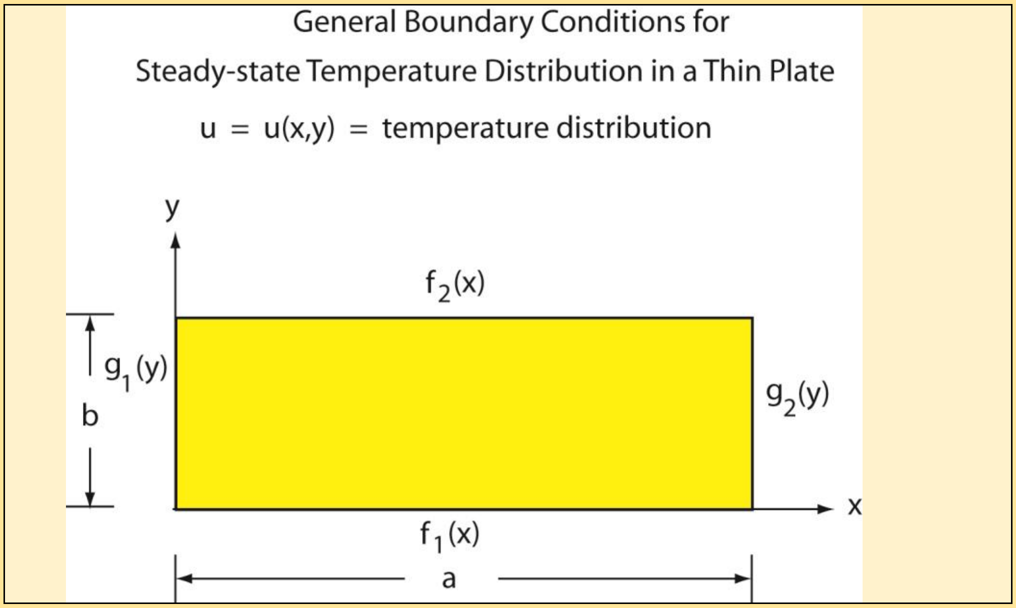
$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0 \quad \text{-------------------------------------- } (2)$$

Strategy: Use the method of separation of variables to solve (2) subject to the boundary conditions.

Consider a thin plate with dimensions (a by b) shown below. The objective is to find the steady state temperature distribution given boundary conditions on each edge of the plate.



A complicated case exists when each edge has a nonhomogeneous boundary condition. i.e. $u(x,0) = f_1(x)$, $u(x,b) = f_2(x)$ $(0 < x < a)$ and
$u(0,y) = g_1(y)$, $u(a,y) = g_2(y)$ $(0 < y < b)$ See the figure below.



In each case the equation for steady state heat conduction is $u_{xx} + u_{yy} = 0$.

Strategy: Split the original problem into four problems each with one nonhomogeneous boundary condition as in the table below. Then use separation of variables for each problem. The sum of the four solutions is then the solution of the original problem.

Prob 1: $u(0,y) = u(a,y) = 0$ $u(x,b) = 0$, $u(x,0) = f_1(x)$	Solution: $u_1(x,y)$
Prob 2: $u(0,y) = 0$, $u(a,y) = g_2(y)$, $u(x,0) = u(x,b) = 0$	Solution: $u_2(x,y)$
Prob 3: $u(0,y) = g_1(y)$, $u(a,y) = 0$, $u(x,0) = u(x,b) = 0$	Solution: $u_3(x,y)$
Prob 4: $u(x,0) = 0$, $u(x,b) = f_2(x)$ $u(0,y) = u(a,y) = 0$	Solution: $u_4(x,y)$

The general solution, $u(x,y)$, for this complicated case is then:

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

In a Nut Shell: The typical heat conduction problem involves finding the steady-state temperature distribution in the plate subject to specified boundary conditions on each edge of the plate. The various boundary conditions include:

The edges of the plate have specified temperatures. In that case:

$u(0,y) = f_1(y)$, $u(a,y) = f_2(y)$ = prescribed temperature at edges $x = 0$ and $x = a$

$u(x,0) = h_1(x)$, $u(x,b) = h_2(x)$ = prescribed temperature at edges $y = 0$ and $y = b$

The edges of the plate might be insulated. In that case:

$u_x(0,y) = 0$, $u_x(a,y) = 0$ = insulated edges at $x = 0$ and $x = a$

$u_y(x,0) = 0$, $u_y(x,b) = 0$ = insulated edges at $y = 0$ and $y = b$

Note: The actual boundary conditions might involve any combination of these b.c's. However, you always need to have a total of four boundary conditions since the governing equation, $u_{xx} + u_{yy} = 0$, has second derivatives in both x and y.

The heat conduction equation involves two independent variables, x and y.
Strategy: Use "separation of variables" to separate out the spatial variables, x and y.

Assume $u(x,y) = X(x)\,Y(y)$ -- (3)

Put this expression into $u_{xx} + u_{yy} = 0$, eq (2), and take derivatives.

$$X''Y + XY'' = 0$$

Next, separate the variables. (division by XY)

$$-\,X''/X = Y''/Y = \text{separation constant} = \lambda$$

So $$X'' + \lambda X = 0$$

and $$Y'' - \lambda Y = 0$$

Note: The separation constant, λ, can take on three possible cases -- such as $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. You need to evaluate each case.

Example: Solve for the steady state heat conduction in a plate (a by b) with edges $x = 0$ and $x = a$ insulated.

Let the temperature along the edge, $y = 0$, be $u(x, 0) = 0$ and the temperature distribution along the edge, $y = b$, is $u(x, b) = f(x)$. See the figure below.

So $$u_{xx} + u_{yy} = 0 \qquad \text{(eq. 1)}$$

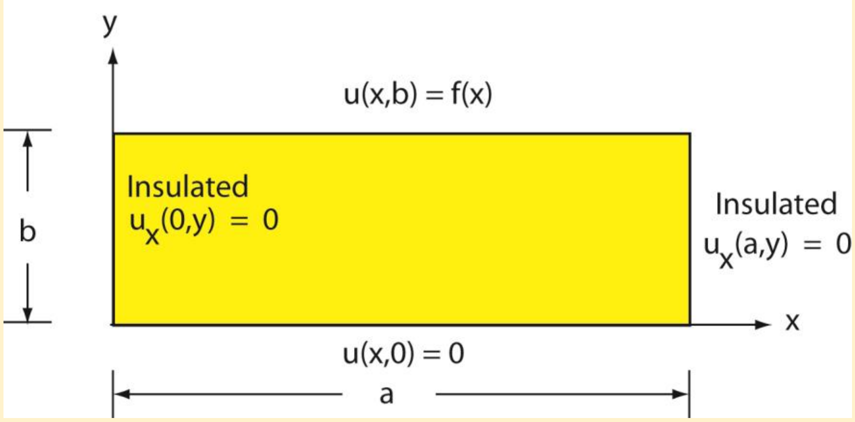
$$u_x(0, y) = u_x(a, y) = 0$$

$$u(x, 0) = 0 \quad \text{and} \quad u(x, b) = f(x)$$

Example

Steady-state Temperature Distribution in a Thin Plate

$u = u(x,y) =$ temperature distribution



Strategy: Separate the variables by assuming $u(x,y) = X(x)Y(y)$, by putting this expression into eq. 1 above, and by dividing by XY.

$$-X''/X = Y''/Y = \text{separation constant} = \lambda$$

So $$X'' + \lambda X = 0$$

and $$Y'' - \lambda Y = 0$$

The separation constant, λ, can take on three possible cases -- such as $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. You need to evaluate each case.

Next consider each case for the separation constant, λ, individually. Recall that the three cases are: $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$.

Note: The boundary conditions at edges of the plate ($x = 0$ and $x = a$) are homogeneous.

$$X'(0) = X'(a) = 0$$

Strategy: Determine the eigenvalues and related eigenfunctions for the eigenvalue problem with homogeneous boundary conditions.

Start with the equation $-X''/X = \lambda$.

For Case 1: $\lambda = 0$ $X'' = 0$, $X(x) = Ax + B$, $X'(x) = A$

So $X'(0) = 0 = A$ and therefore $X(x) = B$, $X_0(x) = 1$

The eigenvalue $\lambda_0 = 0$ and the associated eigenfunction is $X_o(x) = 1$

For Case 2: $\lambda > 0$ $\lambda = \alpha^2$ $X'' + \alpha^2 X = 0$

$$X(x) = A\cos\alpha x + B\sin\alpha x$$

$$X'(x) = -A\alpha\sin\alpha x + B\alpha\cos\alpha x$$

$X'(0) = 0 = B\alpha$, since $\alpha \neq 0$, $B = 0$

$X'(a) = 0 = -A\alpha\sin a\alpha$ and $\alpha \neq 0$ with $A \neq 0$ for a nontrivial solution

Thus $\sin a\alpha = 0$, so $a\alpha = n\pi$ $n = 1, 2, 3, \ldots$

The eigenvalues are $\lambda_n = \alpha_n^2 = (n\pi/a)^2$ with eigenfunctions $X_n(x) = \cos(n\pi x/a)$

For Case 3 $\lambda < 0$ $\lambda = -\alpha^2$ $X'' - \alpha^2 X = 0$

$$X(x) = A\cosh\alpha x + B\sinh\alpha x$$

$$X'(x) = A\alpha\sinh\alpha x + B\alpha\cosh\alpha x$$

$X'(0) = 0 = B\alpha$, since $\alpha \neq 0$, $B = 0$ and $X(x) = A\cosh\alpha x$

$X'(a) = 0 = A\alpha\sinh a\alpha$ and since $\sinh a\alpha$ and $\alpha \neq 0$, $A = 0$

which yields the **trivial solution for Case 3.**

Therefore there are no eigenvalues nor eigenvectors for this case.

Next continue with the solution for $Y_n(y)$. **Strategy:** Use the eigenvalues already determined.

For Case 1 $\lambda = 0$ $Y''(y) = 0$

$Y(y) = Ay + B$ and $Y(0) = 0 = B$

So for $\lambda_0 = 0$ $Y_0(y) = y$

Now **for Case 2** $\lambda_n = \alpha_n^2 = (n\pi/a)^2$ $Y''(y) - (n\pi/a)^2 Y(y) = 0$

with the general solution $Y(y) = A\cosh(n\pi y/a) + B\sinh(n\pi y/a)$

Now with $Y(0) = 0 = A$, $Y_n(y) = \sinh(n\pi y/a)$

Result: The set of eigenvalues and eigenfunctions for cases 1 and 2 are:

$\lambda_0 = 0$, $X_0 = 1$, $Y_0 = y$, $\lambda_n = (n\pi/a)^2$, $X_n = \cos(n\pi x/a)$, $Y_n = \sinh(n\pi y/a)$

Strategy: Sum the "product" solutions: $u(x,y) = u_0(x,y) + \sum u_n(x,y)$

where $u_0(x, y) = X_0 Y_0 = (1)(y) = y$

and $u_n(x, y) = X_n Y_n = \cos(n\pi x/a) \sinh(n\pi y/a)$

$$\text{So} \quad u(x, y) = b_o y + \sum_{n=1}^{\infty} b_n \sinh(n\pi y/a) \cos(n\pi x/a)$$

Next satisfy the temperature distribution, f(x), along $y = b$

$$\text{So} \quad u(x, y) = b_o y + \sum_{n=1}^{\infty} b_n \sinh(n\pi y/a) \cos(n\pi x/a) \quad \text{------------------} \quad (1)$$

The temperature distribution along $y = 0$ is $u(x,b) = f(x)$.

$$\text{So} \quad u(x, b) = f(x) = b_o b + \sum_{n=1}^{\infty} b_n \sinh(n\pi b/a) \cos(n\pi x/a) \quad \text{---------} \quad (2)$$

Strategy: Represent f(x) with a Fourier cosine series as follows:

(Since the x-dependence of u(x, y) depends on $\cos(n\pi x/a)$)

$$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/a) \quad \text{--------------------------------} \quad (3)$$

Strategy: Determine b_o and b_n by comparing coefficients between (2) and (3).

$b_o b = a_0/2$ and $b_n \sinh(n\pi b/a) = a_n$

So $b_o = a_0/2b$ and $b_n = a_n / \sinh(n\pi b/a)$

Thus the solution for the steady state temperature distribution in the plate is:

$$u(x, y) = a_0 y/2b + \sum_{n=1}^{\infty} a_n [\sinh(n\pi y/a) / \sinh(n\pi b/a)] \cos(n\pi x/a)$$

where

$$a_n = (2/a) \int_0^a f(x) \cos(n\pi x/a)dx \quad \text{and} \quad a_0 = (2/a) \int_0^a f(x)dx$$

Heat Conduction in a Circular or Semi-Circular Plate

In a Nut Shell: The governing equation for heat conduction in a circular or semi-circular plate is:

$$\partial u/\partial t = k [\partial^2 u/\partial r^2 + (1/r) \partial u/\partial r + (1/r^2) \partial^2 u/\partial \theta^2] \text{ ----------------------- } (1)$$

where $u = u(r,\theta,t)$ = the temperature in the plate at any time t
r,θ = the location in the plate
t = the time at which the temperature at x is $u(r,\theta,t)$
and
k is the thermal diffusivity of the material

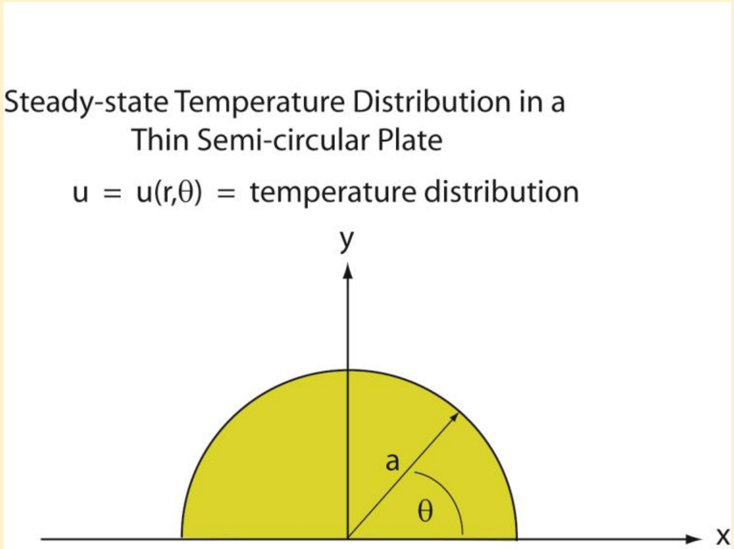
When applied to a plate, the desired outcome is to predict the temperature distribution, $u(r,\theta,t)$, in the plate as a function of time.

For steady-state heat conduction, $\partial u/\partial t = 0$. So the steady-state temperature distribution in the plate is governed by Laplace's equation:

$$\partial^2 u/\partial r^2 + (1/r) \partial u/\partial r + (1/r^2) \partial^2 u/\partial \theta^2 = 0 \text{ ----------------------- } (2)$$

Use the method of separation of variables to solve (2) subject to the boundary conditions.

Consider a thin semi-circular plate of radius a shown below. The objective is to find the steady state temperature distribution given boundary conditions on the boundary of the plate.



The typical heat conduction problem involves finding the steady-state temperature distribution in the semi-circular plate subject to specified boundary conditions on each edge of the plate. The various boundary conditions include:

$$u(a,\theta) = f(\theta), \quad u(r,0) = u(r,\pi) = 0$$

$$u(a,\theta) = f(\theta), \quad u_\theta(r,0) = u_\theta(r,\pi) = 0$$

$$u(a,\theta) = f(\theta), \quad u(r,0) = u_\theta(r,\pi) = 0$$

$$u(a,\theta) = f(\theta), \quad u_\theta(r,0) = u(r,\pi) = 0$$

here $u(a,\theta) = f(\theta)$ is prescribed temperature on $r = a$

In addition the for continuity a finite temperature must exist at $r = 0$ for any θ .

Since the heat conduction equation involves two independent variables, x and y, "separation of variables" is needed to separate out the spatial variables, x and y.

Assume $u(r,\theta) = R(r)\,\theta(\theta)$ --- (3)

Put this expression into eq (2) and take derivatives.

$$R''\theta + (1/r)\,R'\,\theta + (1/r^2)R\theta'' = 0$$

Next, separate the variables. (division by XY)

$$(r^2 R'' + rR') / R = -\theta '' / \theta = \text{separation constant} = \lambda$$

So $r^2 R'' + rR' - \lambda R = 0$

and $\theta '' + \lambda \theta = 0$

The separation constant, λ, can take on three possible cases -- such as

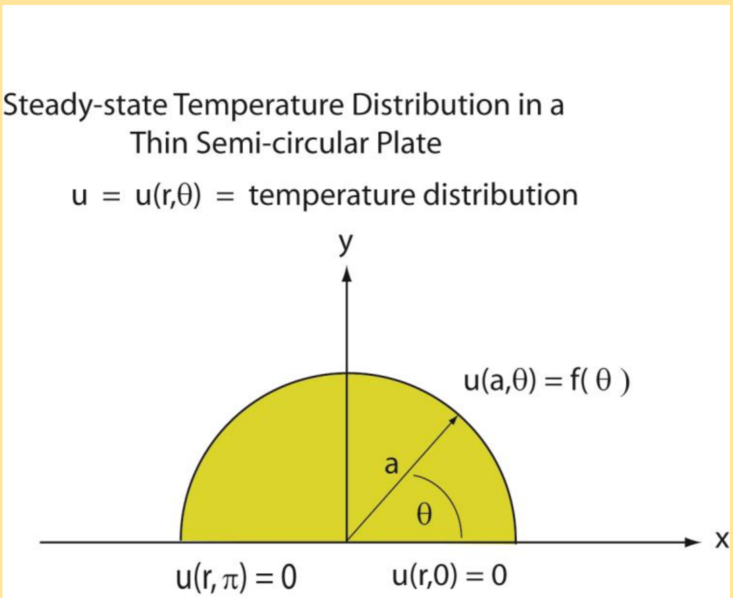
$\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. You need to evaluate each case.

Example: Consider steady state heat conduction in a semicircular plate of radius a shown below. The temperature along the edge, $y = 0$, $u(x, 0) = 0$ and the temperature distribution along the edge, $y = b$, is $u(x, b) = f(x)$. See the figure below.

$$R''\theta + (1/r) R' \theta + (1/r^2)R\theta'' = 0 \qquad (1)$$

$$u(r, 0) = u(r, \pi) = 0$$

$u(a, \theta) = f(\theta)$ (prescribed temperature distribution on r = a)



Strategy: Separate the variables by assuming $u(r,\theta) = R(r)\,\theta(\theta)$, by putting this expression into eq. 1 above. The result is:

$$(r^2 R'' + rR')/R = -\theta''/\theta = \text{separation constant} = \lambda$$

So $$r^2 R'' + rR' - \lambda R = 0$$

and $$\theta'' + \lambda\theta = 0$$

The separation constant, λ, can take on three possible cases -- such as $\lambda = 0$, $\lambda > 0$, and $\lambda < 0$. You need to evaluate each case.

Note that the boundary conditions are: $\theta(0) = \theta(\pi) = 0$

Strategy: Start with the eigenvalue problem: $\theta'' + \lambda\theta = 0$

and examine the three possible cases for the eigenvalues of λ as shown in the table below.

Case 1: $\lambda = 0$ $\quad \theta'' = 0$ or $\quad \theta(\theta) = A\theta + B$

So $\theta(0) = B$ and $\theta(\pi) = A\pi = 0$ So $A = 0$

Result: There are no eigenvalues for this case.

Case 2: $\lambda < 0$ $\quad \lambda = -\alpha^2$ and $\theta'' - \alpha^2\theta = 0$

$$\theta(\theta) = A\cosh\alpha\theta \;\; B\sinh\alpha\theta$$

$$\theta(0) = 0 = A \quad \text{and} \quad \theta(\pi) = 0 = B\sinh\alpha\pi$$

Since $\alpha \neq 0$ and $\sinh\alpha\pi \neq 0$ Therefore

$B = 0$ and there are no eigenvalues for this case.

Case 3: $\lambda > 0$ $\quad \lambda = \alpha^2$ and $\theta'' + \alpha^2\theta = 0$

$$\theta(\theta) = A\cos\alpha\theta + B\sin\alpha\theta$$

$$\theta(0) = A \quad \text{and} \quad \theta(\pi) = 0 = B\sin\alpha\pi$$

For a nontrivial solution $B \neq 0$. So $\sin\alpha\pi = 0$ and $\alpha_n\pi = n\pi$

Therefore $\alpha_n = n$ and the eigenvalues for this case are $\lambda_n = n^2$

with associated eigenfunctions $\theta_n = \sin n\theta$

So $\qquad r^2 R'' + rR' - n^2 R = 0$

Note that this differential equation has variable coefficients. So assume

$R(r) = r^K$ and substitute into the differential equation .

Note: $R' = k\, r^{K-1}$ and $R'' = k(k-1)\, r\, r^{K-2}$

So $\qquad [\, k(k-1) + k - n^2\,]\, r^K = 0$

Sine $r^K \neq 0 \qquad k^2 - n^2 = 0$ and $k = \pm\, n$ which yields

$R(r) = C\, r^n + D r^{-n}$ and for a continuous solution at $r = 0$ D must be zero.

Therefore

$R_n(r) = C_n\, r^n$ and $u_n(r, \theta) = R_n(r)\, \Theta_n(\theta)$

So $\quad u((r, \theta) = \sum_{n=1}^{\infty} R_n(r)\, \Theta_n(\theta) = \sum_{n=1}^{\infty} C_n\, r^n \sin n\theta$

Now $\quad u(a,\theta) = f(\theta) = \sum_{n=1}^{\infty} C_n\, a^n \sin n\theta = \sum_{n=1}^{\infty} b_n \sin n\theta$

where $b_n = (2/\pi) \int_0^{\pi} f(\theta) \sin n\theta \; d\theta$

By comparing terms $C_n\, a^n = b_n$

So $\qquad C_n = b_n/a^n = (2/\pi\, a^n) \int_0^{\pi} f(\theta) \sin n\theta\, d\theta$

and finally

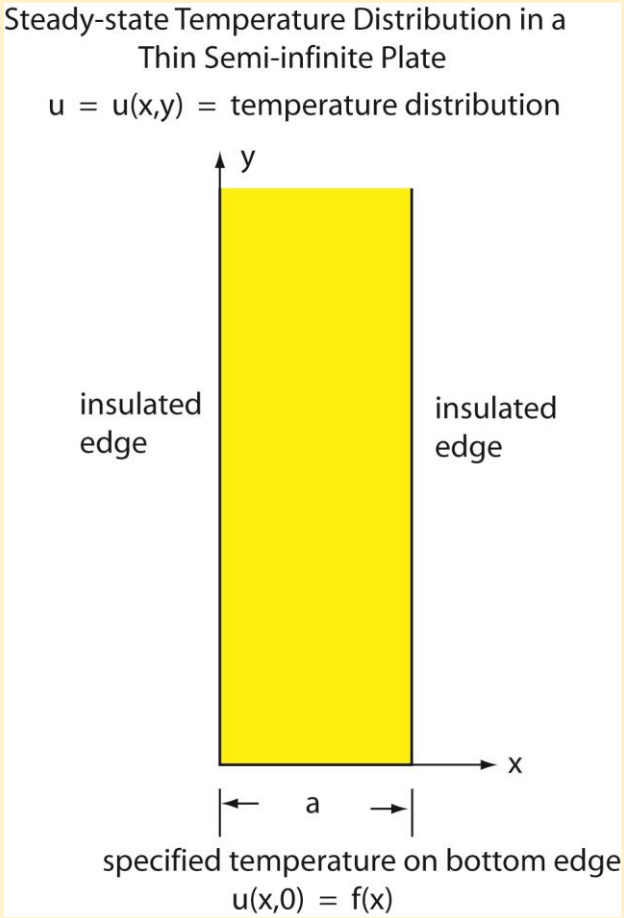
$$u(r,\theta) = \sum_{n=1}^{\infty} C_n\, r^n \sin n\theta$$

Example Heat Conduction in a Semi-Infinite Plate

Consider steady state heat conduction in the semi-infinite plate shown below with edges x = 0 and x = a insulated and with the temperature distribution along the bottom edge of the plate also specified. Solve for the temperature distribution in the plate.

So $\qquad u_{xx} + u_{yy} = 0$ ----------------------------------- (1)

$u_x(0, y) = u_x(a, y) = 0$ and $u(x, 0) = f(x)$



Separate variables by assuming $u(x,y) = X(x)Y(y)$ and by putting this expression into eq. 1 above. Then divide by XY.

$$-X''/X = Y''/Y = \text{separation constant} = \lambda$$

So $$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0$$

The separation constant, λ, can take on three possible cases -- such as $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. You need to evaluate each case.

Note that the boundary conditions are :

$$X'(0) = X'(a) = 0$$

So start with the equation $X'' + \lambda X = 0$

So for Case 1 $\lambda = 0$ $X'' = 0$ or $X(x) = Ax + B$

So $X'(0) = 0 = A$ and $X'(a) = 0$, So $X(x) = B$

and $\lambda_o = 0$ is an eigenvalue with associated eigenvector $X_o(x) = B$

For Case 2 $\lambda < 0$ $\lambda = -\alpha^2$ and $\theta'' - \alpha^2\theta = 0$

$X(x) = A\cosh \alpha x + B\sinh \alpha X$ $X'(0) = 0 = B\alpha$ and $\alpha \neq 0$ so $B = 0$

$X'(a) = 0 = A\alpha \sinh \alpha a$ Since $\alpha \neq 0$ and $\sinh \alpha a \neq 0$

Therefore $A = 0$ and there are no eigenvalues associated with this case.

For Case 3 $\lambda > 0$ $\lambda = \alpha^2$ $\theta'' + \alpha^2\theta = 0$

$X(x) = A\cos \alpha x + B\sin \alpha X$

$X'(0) = 0 = B\alpha$ Since $\alpha \neq 0$ $B = 0$

and $X'(a) = 0 = -A\alpha \sin \alpha a$

For a nontrivial solution $A \neq 0$. So $\sin \alpha a = 0$ and $\alpha_n a = n\pi$

Therefore $\alpha_n = n\pi/a$ and the eigenvalues for this case are $\lambda_n = n^2\pi^2 / a^2$

with associated eigenfunctions $X_n(x) = \cos n\pi x/a$

Next continue with the solution for $Y_n(y)$ using the eigenvalues already determined.

For Case 1 $\lambda = 0$ $Y''(y) = 0$

$Y(y) = Cy + D$ and $C = 0$ for $Y(y)$ to be bounded as $y \rightarrow \infty$

So for $\lambda_o = 0$ $y_o(y) = D$

and $u_o(x,y) = x_o(x)y_o(y) = (B)(D) = C_o$

Now **for Case 3** $\lambda_n = \alpha_n^2 = (n\pi/a)^2$ $Y''(y) - (n\pi/a)^2\, Y(y) = 0$

With the general solution $Y(y) = C\exp[\, n\pi y/a\,] + D\exp[\,-n\pi y/a]$

Now with $\lim_{y\to\infty} Y(y)$ bounded C must be zero.

So $$Y_n(y) = \exp[-n\pi y/a]$$

So $$u_n(x, y) = X_n Y_n = \cos(n\pi x/a)\,[\exp(-n\pi y/a)]$$

So $$u(x, y) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\pi x/a)\,[\exp(-n\pi y/a)]$$

Now $$u(x, 0) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\pi x/a) = f(x)$$

Choose $C_o = a_o/2$ and $C_n = a_n$ where the Fourier coefficients are

$$a_o = (2/a)\int_0^a f(x)\,dx \quad \text{and} \quad a_n = (2/a)\int_0^a f(x)\cos(n\pi x/a)\,dx$$

And the temperature distribution becomes

$$u(x, y) = a_o/2 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/a)\,[\exp(-n\pi y/a)]$$