Vector Fields

In a Nut Shell: Functions that assign vectors in a plane or in space are termed vector fields. You are familiar with vectors such as the unit vectors **i, j, k** along the x, y, and z-axes.

A vector field in a plane takes the form $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ where **P(x,y) and Q(x,y) are scalar fields**. (scalar function). Here P(x,y) is the component of **F**(x,y) in the x-direction and Q(x,y) is the component of **F**(x,y) in the y-direction.

A vector field can also be defined by a vector valued function at each point (x,y,z) in space such as

$$\mathbf{F}(x, y, z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$$

where **P(x,y,z), Q(x,y,z), and R(x,yz) are scalar fields** (scalar function)

There are several theorems dealing with vector fields that are important in your study of multi-variable calculus. So it is important that you understand and can work with vector fields.

The Gradient of a scalar field, Grad F, **is defined as follows:**

in a plane $\quad \text{Grad } F = \partial F/\partial x\,\mathbf{i} + \partial F/\partial y\,\mathbf{j}$

in space $\quad \text{Grad } F = \partial F/\partial x\,\mathbf{i} + \partial F/\partial y\,\mathbf{j} + \partial F/\partial z\,\mathbf{k}$

Definition of the Divergence of a Vector Field, F, div **F**

in a plane $\quad \mathbf{F}(x, y, z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)$

$$\text{div } \mathbf{F} = \partial P/\partial x\,\mathbf{i} + \partial Q/\partial y\,\mathbf{j}$$

in space $\quad \mathbf{F}(x, y, z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$

$$\text{div } \mathbf{F} = \partial P/\partial x\,\mathbf{i} + \partial Q/\partial y\,\mathbf{j} + \partial R/\partial z\,\mathbf{k}$$

Curl of a Vector Field in a plane, curl **F**, where $\mathbf{F} = \mathbf{F}(x, y)$

In general: $\mathbf{F}(x, y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$

$$\text{curl } \mathbf{F} = (\partial/\partial x\ \mathbf{i} + \partial/\partial y\ \mathbf{j}) \times (P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}) \quad \text{(cross product)}$$

$$\text{curl } \mathbf{F} = (\partial Q/\partial x - \partial P/\partial y)\,\mathbf{k}$$

Curl of a Vector Field in space, curl **F** , where **F** = **F**(x, y, z)

In general: **F**(x, y, z) = P(x,y,z) **i** + Q(x,y,z) **j** + R(x,y,z) **k**

$$\text{curl } \mathbf{F} = \det \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{matrix}$$

where det means determinant.

Expansion of this determinant gives for the curl of the vector field, **F**(x,y,z)

$$\text{curl } \mathbf{F} = (\partial R/\partial y - \partial Q/\partial z)\, \mathbf{i} + (\partial P/\partial z - \partial R/\partial x)\, \mathbf{j} + (\partial Q/\partial x - \partial P/\partial y)\, \mathbf{k}$$

Note: A vector field may be conservative or non-conservative. If the curl of the vector field is zero, then the vector field is conservative.

A conservative vector field is said to be irrotational.

A common application appears in the area of fluid mechanics.

Example of a vector field in a plane, (x,y): **F** = [sin x] **i** + [cos y] **j**

Example of a vector field in space, (x,y,z):

$$\mathbf{F} = [\sin x + \exp(y)]\, \mathbf{i} + [\ln x - 3y + z^2]\mathbf{j} + [xyz]\, \mathbf{k}$$

Example of the Gradient of a scalar field, $F = 3x^2 - 2y + 1/z$

Then $\text{Grad } F = \partial F/\partial x\, \mathbf{i} + \partial F/\partial y\, \mathbf{j} + \partial F/\partial z\, \mathbf{k}$

$\partial F/\partial x = 6x$, $\partial F/\partial y = -2$, $\partial F/\partial z = -1/z^2$ $\text{Grad } F = 6x\, \mathbf{i} - 2\, \mathbf{j} - 1/z^2\, \mathbf{k}$

Example of the Divergence of a Vector Field, F **F** = sin x **i** + cos y **j**

$\text{div } \mathbf{F} = \partial/\partial x\, (\sin x) + \partial/\partial y\, (\cos y)$ $\text{div } \mathbf{F} = \cos x + -\sin y$

Note that this vector field is conservative. i.e. curl **F** = **0**

Example of the Curl of a Vector Field, F

$$\mathbf{F} = [\sin x + \exp(y)]\,\mathbf{i} + [\ln x - 3y + z^2]\,\mathbf{j} + [xyz]\,\mathbf{k}$$

$$\text{curl }\mathbf{F} = \det \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin x + \exp(y) & \ln x - 3y + z^2 & xyz \end{array}$$

where det is a 3x3 determinant

$$\text{curl }\mathbf{F} = [xz - 2z]\,\mathbf{i} - [yz - 0]\,\mathbf{j} + [1/x - \exp(y)]\,\mathbf{k}$$

Conservative Vector Fields

In a Nut Shell: A vector field may be conservative or non-conservative. If the curl of the vector field is zero, then the vector field is conservative.

A conservative vector field is said to be irrotational.
A common application appears in the area of fluid mechanics.

A vector field, F, can be defined by a vector valued function at each point (x,y) in a plane or by each point (x,y,z) in space such as

$$\mathbf{F}(x, y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$$

$$\mathbf{F}(x, y, z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}$$

where P(x,y), Q(x,y), and R(x,y) are scalar fields (scalar functions)
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Strategy to test for Conservative Vector Fields: Calculate the curl of the vector field to determine if it is conservative or not. If the curl of the vector field is zero, then the vector field is conservative and there exists a scalar function, a potential function, such that the gradient of the scalar function equals the vector function.

Test for a two-dimensional vector field, F(x,y): curl (F(x,y)) = 0

Test for a three-dimensional vector field, F(x,y,z): curl (F(x,y,z)) = 0

For conservative vector fields there exists a scalar function such that the vector function equals the gradient of the scalar function.

$$\mathbf{F}(x,y) = \text{grad}\,(\,f(x,y)\,) \qquad \text{(two-dimensional case)}$$

$$\mathbf{F}(x,y,z) = \text{grad}\,(\,f(x,y,z)\,) \qquad \text{(three-dimensional case)}$$

Example: Determine if the following vector field, $\mathbf{F}(x,y)$ is conservative. If it is conservative, then find the scalar, potential function, $f(x,y)$.

$$\mathbf{F}(x,y) = (\, y \sin x\,)\,\mathbf{i} + (\, -\cos x\,)\,\mathbf{j}$$

TEST:

$$\operatorname{curl} \mathbf{F}(x,y) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & 0 \\ y \sin x & -\cos x & 0 \end{vmatrix}$$

$$\operatorname{curl} \mathbf{F}(x,y) = (\,0\,)\,\mathbf{i} + (\,0\,)\,\mathbf{j} + (\,\sin x - \sin x)\,\mathbf{k} = \mathbf{0}$$ Vector field, **F**, is conservative.

Now: $\operatorname{grad} (f(x,y) = \mathbf{F}(x,y) = (\, y \sin x\,)\,\mathbf{i} + (\, -\cos x\,)\,\mathbf{j}$

$\partial f/\partial x = y \sin x$ and $\partial f/\partial y = -\cos x$

Integrate: $f(x,y) = -\, y \cos x + g(y)$

$\partial f/\partial y = -\cos x + dg(y)/dy = -\cos x$ Therefore $dg/dy = 0$

and $g(y) = C =$ constant

So the scalar function $f(x,y) = -\, y \cos x + C$ (result)

Example: Determine if the following vector field, $\mathbf{F}(x,y,z)$ is conservative. If it is conservative, then find the scalar, potential function, $f(x,y,z)$.

$$\mathbf{F}(x,y) = (\, y \sin x\,)\,\mathbf{i} + (\, -\cos x\,)\,\mathbf{j} + (\, z \tan x\,)\,\mathbf{k}$$

TEST:

$$\operatorname{curl} \mathbf{F}(x,y) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial / \partial x & \partial / \partial y & \partial / \partial z \\ y \sin x & -\cos x & \tan x \end{vmatrix}$$

$$\operatorname{curl} \mathbf{F}(x,y) = (\,0\,)\,\mathbf{i} + (\,\sec^2 x\,)\,\mathbf{j} + (\,\sin x - \sin x)\,\mathbf{k} \neq \mathbf{0}$$

Result: The vector field, **F**, is not conservative. So there is no scalar potential function.