Integrals using Integration by Parts

In a Nut Shell: Integration by parts is very useful for integrals of the form $I = \int u\, dv$.

The basic relation used for Integration by Parts is:

$I = \int u\, dv = uv - \int v\, du$ ← This is the basic relation

Example: $\int(\ln x / x^2)dx$

let $u = \ln x$, $dv = (1/ x^2)\, dx$ ← Standard form for integral

Here $\int (1/ x^2)\, dx$ is a standard integral of the form $\int x^n\, dx = (x^{n+1})/(n+1) + c$

So $du = (1/x)dx$ and $v = - (1/x)$

Then I becomes $-(1/x) \ln x + \int(1/x^2)\, du$

with the standard form again $\int x^n\, dx = (x^{n+1})/(n+1) + c$

Result: $I = - (1/x) \ln x - 1/x + c$

c is the constant of integration

Method: Multiple Use of Integration by Parts

$I = \int u\, dv = uv - \int v\, du$

Example: $I = \int x^2 e^{2x}\, dx$

let $u = x^2$, $dv = e^{2x}\, dx$ ← Standard integral $\int e^{ax}\, dx = (1/a) e^{ax} + c$

$du = 2x\, dx$ $v = ½ e^{2x}$

Then I becomes $½ x^2e^{2x} - \int x e^{2x}\, dx$ and integrate by parts again,

let $u = x$, $dv = e^{2x}\, dx$ ← Standard integral $\int e^{ax}\, dx$

$du = dx$ $v = ½ e^{2x}$

Result: $I = ½ x^2e^{2x} - ½ x e^{2x} + ¼ e^{2x} + c$

c is the constant of integration

In a Nut Shell: There are three common types of integrals where integration by parts is the recommended procedure. Each integral has the form $\mathbf{I = \int u\, dv = uv - \int v\, du}$. It is best to identify these types before applying integration by parts. The table below details the strategies for these types along with examples.

Type 1: The integral, I, has the following form: $\int f(x)\, g(x)\, dx$.

Here you know the integrals for both f(x) and g(x). So you need to pick

u and dv such that it simplifies the integral $\int v\, du$ compared with the

original integral $\int f(x)\, g(x)\, dx$.

Example: $I = \int x \cos x \, dx$ Here you know both integrals for x and for cos x.

Pick $u = x$ and $dv = \cos x \, dx$
Then $du = dx$ and $v = \sin x$ so $I = x \sin x - \int \sin x \, dx$

Finally $I = x \sin x + \cos x + C$ (result)

Note: If you were to pick $u = \cos x$ and $dv = x\, dx$ the next integral

Then $du = -\sin x \, dx$ and $v = (x^2 / 2)$ and the second integral

would be more complicated. i.e. $\int ((x^2 / 2) \sin x \, dx$ than the one you

started out with.

Type 2: The integral, I, has the following form: $\int f(x)\, g(x)\, dx$. Here suppose you

only know the integral for one of the two functions, f(x) and g(x). So you need

to pick u for the function where you don't know its integral.

Example: $I = \int x \ln x \, dx$ Here you don't know the integral for ln x.

Therefore pick $u = \ln x$ and $dv = x$

Then $du = (1/x)\, dx$ and $v = (x^2 / 2)$ so $I = (x^2 / 2) \ln x - \int (x/2)\, dx$

Finally $I = (x^2 / 2) \ln x - x^2 /4 + C$ (result)

In a Nut Shell: Type 3 is the most complicated since it involves integration by parts twice.

The integral, I, has the following form: $\int f(x)\, g(x)\, dx$. Here you know the

integrals for both f(x) and g(x). However, you will need to integrate twice in

order to return an integral similar to the original integral. So you need to pick

u and dv such that the integral $\int v\, du$ in the second integration is similar

to the original integral $\int f(x)\, g(x)\, dx$ except for a constant.

Example: $I = \int e^x \cos x \, dx$

Here you know both integrals for e^x and for $\cos x$.

For the first integration:

Pick $u = e^x$ and $dv = \cos x \, dx$
Then $du = e^x \, dx$ and $v = \sin x$ so $I = e^x \sin x - \int e^x \sin x \, dx$

For the second integration:

Pick $u = e^x$ and $dv = - \sin x \, dx$

Then $du = e^x \, dx$ and $v = \cos x$

So $I = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

But $\int e^x \cos x \, dx = I$ **Collect terms:** So $2I = e^x \sin x + e^x \cos x$

And finally $I = (1/2) [e^x (\sin x + \cos x)] + C$ (result)