Introduction to Differential Equations

In a Nut Shell: A differential equation is an equation where at least one term in the equation involves a derivative. Differential equations describe mechanical, electrical, and fluid systems in engineering. The subject of differential equations provides an opportunity to apply math to engineering applications such as heat transfer and vibrations.

The method of solution for a differential equation (d.e.) depends on its type. So you need to be able to classify the type of differential equation.

Classifications include the following terms:

What is the order of the differential equation ?
Is the differential equation an Ordinary or Partial differential equation ?
Is the differential equation. Linear or Nonlinear ?
Does the differential equation have Constant or Variable Coefficients ?
Is the differential equation Homogeneous or Non-homogeneous?
Is the differential equation Separable or non-separable?
Is the differential equation exact?

Let's answer these questions.

The **order of a d.e**. is the order of the highest derivative of the dependent variable in the d.e. In the examples below y is the dependent variable and x is the independent variable. i.e.

$dy/dx = 5$ is a first order d.e. (due to the term dy/dx)

$d^2y/dx^2 + dy/dx + y = 3x$ is a second order d.e. (due to the term d^2y/dx^2)

A **differential equation is classified as ordinary** if it contains only one independent variable, x. The two examples in the table above are both ordinary differential equations.

A differential equation that has more than one independent variable, say both x and t, is a **partial differential equation.**

i.e. The differential equation given below is a partial differential equation involving the two independent variables, x and t.

$\partial u/\partial t = k\, \partial^2 u/\partial x^2$ here $u(x,t)$ is the dependent variable. It might represent the

temperature distribution in a rod as a function of position, x, along the rod and the time, t; k is a constant

The general, first order, ordinary differential equation, $\mathbf{y' + p(x)\, y = q(x)}$ is said to be **linear if both its dependent variable y and its derivative y' are linear. (It can have no powers of y or y ' or other nonlinear terms)** Here $y' = dy/dx$.

$y' = 3x^2$ is a linear, first order d.e.

$y' + y^2 = 3x^4$ is a nonlinear, first order d.e. (because of the term y^2)

If $p(x)$ is a constant in the above d.e., then the d.e. has a **constant coefficient**. If $p(x)$ is some function of x, say sin x, then the d.e. has a variable coefficient.

If $q(x)$ in the above d.e. is zero, then the d.e. is said to be **homogeneous**. Else it is **nonhomogeneous**. An example of an nonhomogeneous d.e. occurring in vibrations is:

$$d^2y(t)/dt^2 + y(t) = f(t) = 3\sin t$$

The dependent variable, y, represents displacement and the independent variable, t, represents time.

This d.e. is a second order, linear, ordinary, nonhomogeneous d.e. with constant coefficients. It is nonhomogeneous since $f(t) \neq 0$. In fact, $f(t) = 3 \sin t$ is said to be the "forcing" function", $f(t)$, in a vibration problem

Other examples of second order d.e.'s: (All describe vibration problems such as in suspension systems or various mechanical systems.)

$$d^2y(t)/dt^2 + y(t) = f(t) = 0$$

This d.e. is a second order, linear, ordinary, homogeneous d.e. with constant coefficients. It is an example of a "free vibration". i.e. No forcing function on right hand side, meaning $f(t) = 0$.

$d^2y(t)/dt^2$	$+ 3dy(t)/dt$	$+ 16y(t)$	$=$	$f(t)$	$= 0$
acceleration term	damping term	spring term		forcing function	(in above d.e.)

This d.e. is a second order, linear, ordinary, homogeneous d.e. with constant coefficients. It is an example of a free vibration ($f(t) = 0$) with damping ($3\, dy/dt$). The displacement, $y(t)$, will eventually die out because of system damping.

Separable or Non-separable d.e.'s

A first order d.e. is said to be separable if one can separate the dependent variable, y,

from the independent variable, x, on each side of the equal sign. i.e.

$dy/dx = x \ln x$ is separable ; $dy = x \ln x \; dx$ (separated form)

$dy/dx = x / y$ is separable; $y\, dy = x\, dx$ (separated form)

$dy/dx = x - y$ is non separable (can't separate)
which can be written as $dy/dx + y = x$ (in standard form)

$dy/dx = x^2y$ is separable; $dy/y = x^2\, dx$ (separated form)

Exact d.e.'s A d.e. of the form $M(x,y)dx + N(x,y)dy = 0$ is **exact** if $\partial M/\partial y = \partial N/\partial x$

Common applications (models) involving first order separable d.e.'s:
k is a constant include:

$dy/dt = ky$ (natural growth) d.e. is 1st order, linear, separable
y(t) might represent population

$dy/dt = -ky$ (decay) d.e. is 1st order, linear, separable
y(t) might represent life of an isotope

$dT/dt = k(T_o - T)$ (cooling or heating) d.e. is 1st order, linear, separable
T_o might be ambient temperature, T is temperature of coffee in a cup

$A(y)\, dy/dt = -k\sqrt{y}$ (fluid flow in a tank) d.e. is 1st order, nonlinear, separable
A(y) is profile of tank, y is depth of fluid in tank, t is time

Initial Conditions – Constants of integration need to be evaluated in order to find a particular solution of any d.e. for the prescribed initial condition.

For a first order d.e. i.e.($dy/dx = 3 \sin x$) you will need one initial condition (since you will integrate once) to find a particular solution of the d.e.
(Similar to constant of integration for integrals)
An example of an initial condition is: $y(0) = y_o$ where y_o is a constant.

For a second order d.e. i.e.($d^2y/dx^2 = \sin x$) you will need two initial conditions to find a particular solution of the d.e. (since you will need to integrate twice)

An example is: $y'(0) = dy(0)/dx = a$, $y(0) = b$ where a and b are constants.