

## Basics Methods of Integration/Fundamental Theorem

**In a Nut Shell:** Integrals have the form  $I = \int f(x) dx$  where  $f(x)$  is the integrand of the integral. The integrand may be complicated. The key strategy in evaluating any complicated integral is to simplify it into one or more standard integrals where you know each standard integral. To check your result for the integral, take the derivative of your result. It should return the expression of the integrand if your result is correct.

**Therefore you must memorize the Standard Form Integrals - - This is mandatory.**

Standard integrals include integrals involving functions such as polynomials, exponentials, trig functions, combinations of these and others. You must also memorize trig identities given at the end of this section.

Listed below are methods, procedures, and transformations commonly used to simplify complicated integrals into ones with standard form. **Note:** You may apply them separately, in combination, or group terms.

1.	Simple substitution
2.	Use of trig formulas
3.	Combination of substitution and trig formulas
4.	Grouping of terms along with a substitution
5.	Substitution followed by grouping followed by another substitution
6.	Integration by parts
7.	Combination of substitution and integration by parts
8.	Any of the above in combination perhaps multiple times
9.	Manipulation
10.	Intuition using knowledge of derivative

**Use of a simple substitution**  $I = \int (x+1)^2 dx$  let  $u = x+1$ ,  $du = dx$

Then the integral becomes  $\int u^2 du$  which is of the standard form

$$\int x^n dx = (x^{n+1})/(n+1) + c \quad \text{Result: } I = (x+1)^3 / 3 + C$$

**Use of Trig formulas (in this case twice)** Trig formula  $\sin^2 x = (1 - \cos 2x)/2$

$$\cos^2 x = (1 + \cos 2x)/2$$

Example:  $I = \int \sin^4 x dx = \int [(1 - \cos 2x)/2]^2 dx$

$$I = \frac{1}{4} \int [1 - 2\cos 2x + \cos^2 2x] dx$$

$$I = \frac{1}{4} \int [1 - 2\cos 2x + (1 + \cos 4x)/2] dx$$

So  $I = \frac{1}{4} \int [1 - 2\cos 2x] dx + \frac{1}{8} \int (1 + \cos 4x) dx$  which are standard integrals

$$\text{Result: } I = (3/8)x - (1/4)\sin 2x + (1/32)\sin 4x + C$$

### Combination of substitution and trig formulas

$$I = \int \sin^2(x/8) dx \quad \text{Substitution: } w = x/8, \quad dw = dx/8, \quad dx = 8 dw$$

$$I = 8 \int (\sin^2 w) dw \quad \text{then use trig formula} \quad \sin^2 w = (1 - \cos 2w)/2$$

$$I = 4 \int (1 - \cos 2w) dw \quad \text{which are two standard integrals}$$

### Grouping of terms (along with a simple substitution such as $w = \sec x$ )

$$I = \int \sec^2 x \tan x dx = \int \sec x (\sec x \tan x) dx \quad (\text{shows grouping})$$

$$\text{Then use substitution to give} \quad I = \int w dw \quad \text{which is a standard integral}$$

### Substitution followed by grouping followed by another substitution

$$\text{First substitution} \quad w = x/5, \quad dw = dx/5, \quad dx = 5 dw$$

$$I = \int \sec^2(x/5) \tan(x/5) dx = 5 \int \sec w (\sec w \tan w) dw \quad (\text{shows grouping})$$

$$\text{Second substitution} \quad v = \sec w, \quad dv = \sec w \tan w dw$$

$$\text{Then use substitution to give} \quad I = \int v dv \quad \text{which again is a standard integral}$$

**Integration by parts** is very useful for integrals of the form  $I = \int u dv$ .

Then the integral,  $I$ , becomes

$$I = uv - \int v du \quad \text{i.e. Apply to the integral} \quad I = \int x \cos x dx$$

$$\text{Here} \quad u = x \quad \text{and} \quad dv = \cos x dx$$

$$\text{Then} \quad u = x \quad \text{and} \quad dv = \cos x dx$$

$$du = dx \quad \text{and} \quad v = \sin x$$

$$\text{So} \quad I = x \sin x - \int \sin x dx = x \sin x + \cos x + C \quad (\text{result})$$

### Combination of substitution and integration by parts $I = \int x \cos(2x) dx$

$$\text{Let} \quad w = 2x \quad dw = 2 dx, \quad x = w/2, \quad dx = \frac{1}{2} dw$$

$$I = \int (w/2) \cos w (\frac{1}{2}) dw = \frac{1}{4} \int w \cos w dw \quad \text{then use integration by parts.}$$

**Note: In all cases you must use substitutions to express your final result in terms of  $x$ .**

**Trig substitution, followed by manipulation, followed by grouping of terms, followed by a substitution and finally using partial fractions.**

$$I = \int \left[ \sqrt{(x^2 + 9)} / x \right] dx \quad \text{First use the trig substitution } x = 3 \tan \theta$$

Then  $dx = 3 \sec^2 \theta$  and  $(x^2 + 9) = 9 \sec^2 \theta$  so

$$I = \int 3 (\sec \theta) (3 \sec^2 \theta) d\theta / 3 \tan \theta = 3 \int [\sec^3 \theta / \tan \theta] d\theta$$

**Now use manipulation** by multiplying both the numerator and denominator by  $\tan \theta$ .

The result is  $I = 3 \int [\sec^3 \theta \tan \theta / \tan^2 \theta] d\theta$ ; next group terms as follows

$$I = 3 \int [\sec^2 \theta (\sec \theta \tan \theta) / \tan^2 \theta] d\theta$$

Recall  $1 + \tan^2 \theta = \sec^2 \theta$ , then the integral becomes

$$I = 3 \int [\sec^2 \theta (\sec \theta \tan \theta) / (\sec^2 \theta - 1)] d\theta$$

Next use the substitution  $w = \sec \theta$  so the integral becomes

$$I = 3 \int (w^2 dw / (w^2 - 1)) \quad \text{Now divide } w^2 - 1 \text{ into } w^2 \text{ which gives}$$

$$w^2 / (w^2 - 1) = 1 + 1 / (w^2 - 1) = 1 + 1 / (w - 1)(w + 1) \text{ and the result is}$$

$$I = 3 \int dw + 3 \int dw / (w - 1)(w + 1)$$

Next use partial fractions on the second integral to

$$\text{obtain } I = 3 \int dw + 3/2 \int dw / (w - 1) - 3/2 \int dw / (w + 1)$$

The integration yields

$$I = 3w + 3/2 \ln(w - 1) - 3/2 \ln(w + 1) + C \quad \text{but } w = \sec \theta \text{ so}$$

$$I = 3 \sec \theta + 3/2 \ln(\sec \theta - 1) - 3/2 \ln(\sec \theta + 1) + C$$

Now  $\tan \theta = x/3$  so  $\sec \theta = \sqrt{(x^2 + 9)} / 3$  which gives the final result as follows:

$$I = \sqrt{(x^2 + 9)} + 3/2 \ln(\sqrt{(x^2 + 9)}/3 - 1) - 3/2 \ln(\sqrt{(x^2 + 9)}/3 + 1) + C$$

Sometimes integration by intuition using knowledge of the derivative of the function applies. Here is an example.

$$I = \int 2^x dx$$

The integrand is  $2^x$ . Calculate the derivative.

Let  $y = 2^x$ , then  $\ln y = x \ln 2$  and  $(dy/dx) / y = \ln 2$

So  $d(2^x)/dx = y \ln 2 = 2^x \ln 2$  and

$$I = 2^x / \ln 2 + C \quad \text{is the value of the integral}$$

### The Fundamental Theorem of Calculus

**In a Nut Shell:** The Fundamental Theorem of Calculus has two parts. Use it to evaluate an integral or to take the derivative of an integral.

**Part 1:** If  $f$  is continuous on  $[a, b]$  and  $F(x)$  is any antiderivative of  $f(x)$ , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

**Part 2:** If  $f$  is continuous on  $[a, b]$  and  $F(x) = \int_a^x f(t) dt$ ,

then  $F'(x) = dF(x)/dx = f(x)$ , on  $[a, b]$ .

### Application of Part 2 to finding the derivative of an integral

$$F(x) = \int_{u(x)}^{v(x)} f(t) dt,$$

then  $F'(x) = dF(x)/dx = f(v) dv/dx - f(u) du/dx$