Basics of Multiple Integrals and Applications / Double Integrals

In a Nut Shell: The integral under a curve, y(x), gives the area underneath the curve. For a single integral, the differential area, dA, can be represented by $dA = y\,dx$, by $dA = x\,dy$.

For double integrals the differential area, dA, can be represented by $dA = dy\,dx$ or by $dA = dx\,dy$. For areas using double integrals $A = \int\int dy\,dx$ or $A = \int\int dx\,dy$.

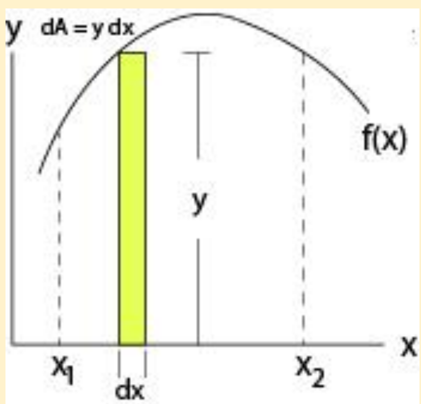
In a similar manner the volume, V, between two surfaces leads to a triple integrals $V = \int\int\int dx\,dy\,dz = \int\int\int dx\,dz\,dy$ or an combination of integration order. Some orders may be easier than others to carry out the integration.

In all cases you need to determine the limits of integration.

Recall that the total area under the curve, y = f(x), in Calculus 2 was given by:

$$A = \int_{x_1}^{x_2} f(x)\,dx = \int_{x_1}^{x_2} y\,dx \text{ ------------------- } (1)$$

The element of area, dA was **visualized as a rectangle** of width dx and height y under the curve y = f(x). The total area then was the "sum" of each rectangle. The region of integration extended from x_1 to x_2 .



Next consider the area between two curves $y_1 = f_1(x)$ and $y_2 = f_2(x)$ where

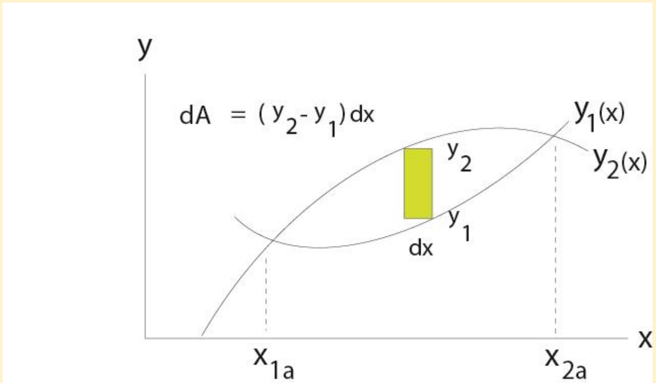
the curve for y_2 lies above y_1 . Using the approach in Calculus 2, the area between

the curves is:

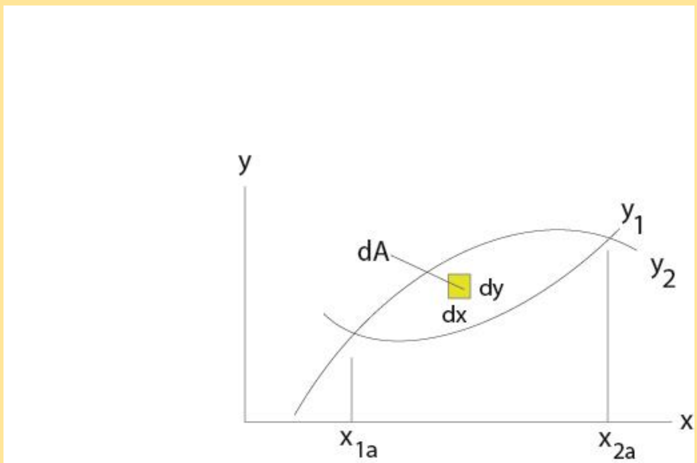
$$A = \int_{x_{1a}}^{x_{2a}} (y_2 - y_1)\,dx \text{ ------------------------------------- } (2)$$

where x_{1a} and x_{2a} are the x-coordinates of the points of intersection of the two curves.

The figure below shows the element of area, dA, for a "single" integral case.



Now consider using a “double” integral for the above case. The element of area dA is given by the small rectangle with dimensions dy by dx. i.e. dA = dy dx

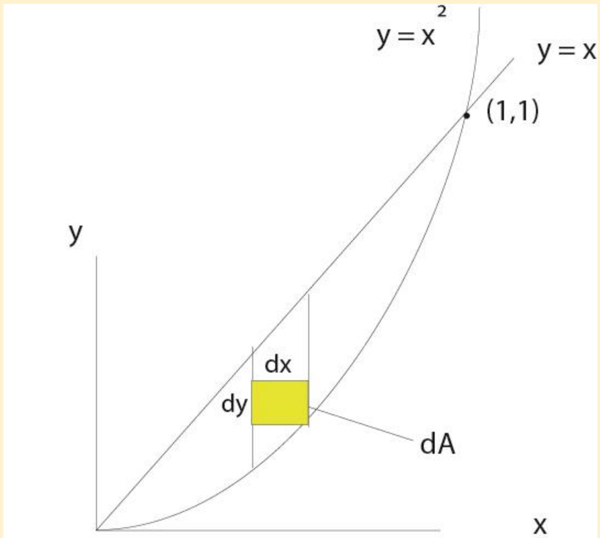


So the total area becomes $A = \int_{x_{1a}}^{x_{2a}} \int_{y_1}^{y_2} dy \; dx$

Here the first integration is on the y-variable. You can picture this as “sweeping” the element of area from y_1 to y_2 followed by “sweeping” the rectangle from x_{1a} to x_{2a} .

The same strategy (but more complicated) applies for calculating volumes between two intersecting surfaces. In this case the element of volume is dV = dx dy dz if you "sweep" the volume first in x, then in y, and finally in z directions.

Example: Find the area bounded by the curves $y = x^2$ and $y = x$ in the first quadrant See the figure below.



Strategy: Draw the element of area (very important) and determine the points of intersection which will provide the limits of integration.

The curve $y = x^2$ lies below $y = x$. The points of intersection are determined by

$$x = x^2 \quad \text{So} \quad x = 0 \text{ and } x = 1$$

Now the element of area is dA where $dA = dx\,dy = dy\,dx$

For the element shown above, choose to integrate on the variable y first.
i.e. Sweep in the "y-direction first. So $dA = dy\,dx$ and the integral becomes:

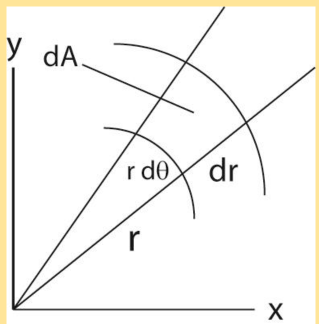
$$A = \int_{x=0}^{x=1} \int_{y=x^2}^{y=x} dy\; dx = \int_{x=0}^{x=1} y \Big|_{x^2}^{x} dx$$

$$A = \int_{x=0}^{x=1} [\, x - x^2 \,]\, dx = [\,(1/2)\,x^2 - (1/3)\,x^3\,] \Big|_0^1 = 1/6$$

In a Nut Shell: The element of area may be expressed in various coordinate systems including rectangular and polar. The table below lists both options.

The element of area dA in rectangular coordinates is $dA = dx\ dy$ or $dA = dy\ dx$.

The element of area dA in polar coordinates has edges $r\,d\theta$ and dr. See the figure below. So the element of area dA in polar coordinates is $dA = r\,d\theta\,dr = r\,dr\,d\theta$



Example: Convert the following integral to polar coordinates. Then evaluate the integral.

$$I = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{(1-y^2)}} \sin(x^2 + y^2)\ dy\ dx$$

From the upper limit on x: $x^2 + y^2 = 1$ so in polar coordinates $r = 1$ and the transformation to polar coordinates is $x = \cos\theta$ and $y = \sin\theta$

If $y = 0$, then $\theta = 0$ (lower limit) and if $y = 1$ $\theta = \pi/2$ (upper limit)

The integral in polar form becomes:

$$I = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=1} \sin(r^2)\ r dr\ d\theta$$ To evaluate this integral let $u = r^2$

then $du = 2\,r\,dr$ or $r\,dr = (1/2)\,du$ and $I = \int_{\theta=0}^{\theta=\pi/2} (1/2) \int_{u=0}^{u=1} \sin u\ du\ d\theta$

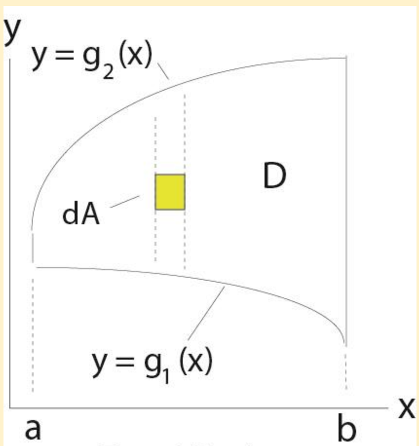
$$I = (1/2) \int_{\theta=0}^{\theta=\pi/2} -\cos u \Big|_0^1 \ d\theta = \pi\,[\,1 - \cos(1)\,]/4$$ (result)

Basics of Double Integrals

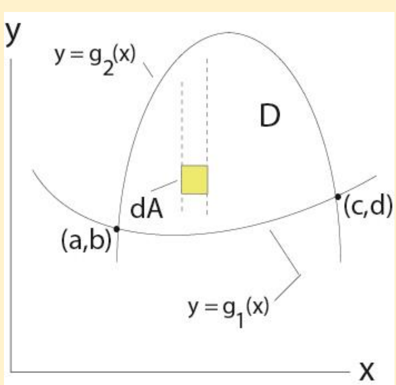
In a Nut Shell: The **single integral of a function**, y(x), $\int y\,dx$ gives the area under the function y(x). Likewise the integral of a function, x(y), $\int x\,dy$ also gives the area under the function, x(y). In these cases the element of area, $dA = y\,dx$ or $dA = x\,dy$.

In a Nut Shell: For double integrals, the element of area, dA, can also be represented by $dA = dx\,dy$ or by $dA = dy\,dx$. This representation gives rise to a "double integral". The value of the integral then becomes $A = \int\int dx\,dy$ or $A = \int\int dy\,dx$ depending on the "order" of integration. Depending on the function or functions one order (i.e. integrate x first followed by y or vice-versa) of integration may be easier than the other order. The order of integration is optional.

The most convenient order of integration depends on the type of region involved. A **Type 1 region** occurs when integration with respect to the y-coordinate comes first followed by integration in the x-direction. The figures below show Type 1 regions. You can think of the process as "sweeping out" the entire domain using the element of area, dA. Here you "sweep" the entire domain, D, first in the y-direction followed by the x-direction.

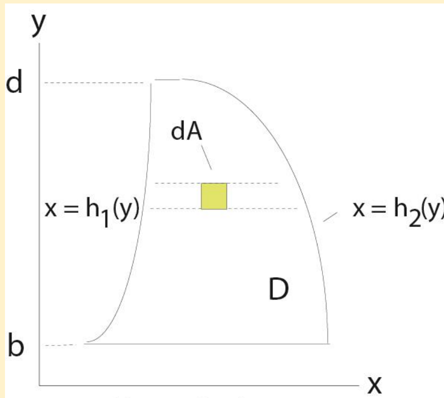


Type 1 Region

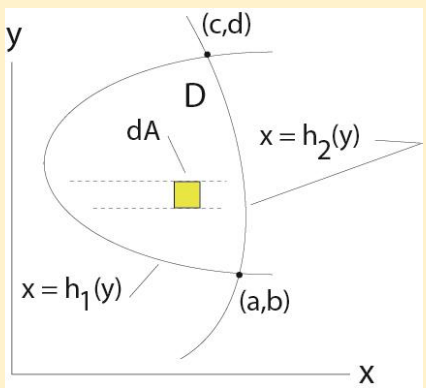


Type 1 Region
Intersecting Functions

A **Type 2 region** occurs when integration with respect to the x-coordinate comes first followed by integration in the y-direction. The figures below show Type 2 regions. You can think of the process as "sweeping out" the area using the element of area, dA. Here you "sweep" the entire domain, D, first in the x-direction followed by the y-direction.



Type 2 Region



Type 2 Region
Intersecting Functions

Strategy recommended for evaluation of double integrals is as follows.

a. Identify the type of Region. Recall:

b. Integrate first in the y-direction for Type 1 Regions and in the x-direction for Type 2 Regions.

c. Determine the limits of integration then integrate. (Type 1, Type 2 shown below.)

d. Evaluate

Type 1

$$A = \int_{x=a}^{x=b} \int_{y = g_1(x)}^{y= g_2(x)} dy\, dx$$

Type 2

$$A = \int_{y=b}^{y=d} \int_{x = h_1(y)}^{x = h_2(y)} dx\, dy$$

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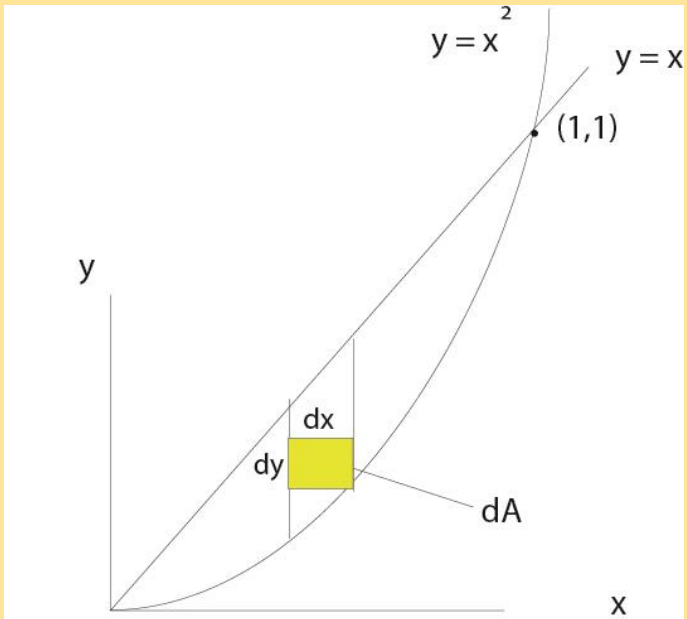
in the first quadrant See the figure below.



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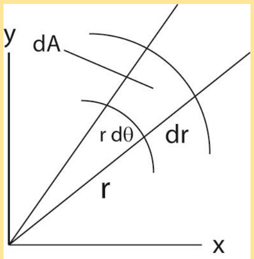

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