Separable Differential Equations/ Growth and Decay Applications

In a Nut Shell: In general, differential equations where the independent variable, t, does not appear explicitly such as:

$$dy/dt = f(y)$$

are termed **"autonomous differential equations".** Mathematical models representing growth and decay fall into this category. The table below gives two of the simpler ones.

Differential equation involving "growth"

$dy/dt = K y$; here rate of change of y is proportional to itself

subject to $y(0) = y_o$ which represents the initial condition of y at $t = 0$

K = proportionality constant

Applications: a. population growth
b. annual rate (income)

$dy/y = K\,dt$ (separate variables y and t and integrate)

$\ln y = K t + C_1$, C_1 is the constant of integration

$y = \exp(Kt + C_1) = C e^{Kt}$

Differential equation involving "decay" (negative growth)

$dy/dt = -K y$; here rate of change of y is proportional to its

subject to $y(0) = y_o$ which represents the initial condition of y at $t = 0$

K = proportionality constant

i.e.

K = decay constant
K = sales decay constant
K = drug elimination constant

Applications: a. radioactive decay (radiocarbon dating)
b. annual rate (income)

$dy/y = -K\,dt$ (separate variables y and t and integrate)

$\ln y = -K t + C_1$, C_1 is the constant of integration

$y = \exp(-Kt + C_1) = C e^{-Kt}$

Example: An initial sample contains 100 cells and grows at a constant rate. After 8 hours there are 2000 cells. Determine the growth rate, find an expression, y(t), for the population of cells at any time t, and determine the number of cells after 24 hours.

Strategy: Apply

$$y(t) = C e^{kt}$$

where $y(t)$ is the size of the population at any time t
$y(0) = 100 = C$ = initial population
k is the growth rate (to be found)

$$y(8) = 100 e^{8k} = 2000$$

$e^{8k} = 20$ So $8k = \ln(20)$ and $k = (1/8) \ln(20)$

$k = \ln(20^{1/8})$ (result)

So $y(t) = 100 \exp[(\ln[20^{1/8}) t]) = (100) 20^{t/8}$ (result)

$y(24) = (100) 20^3 = 800{,}000$ cells (result)

Example: A sample of 50 cancer cells are treated with radiation such that half of them are killed in 28 days. Assume a constant decay rate. Find the decay rate, the general expression for the number of cancer cells after t days, and how many cells still survive after 40 days.

Strategy: Apply $y(t) = C e^{kt}$ $y(0) = 50$

$$y(t) = 50 e^{kt}$$

$$25/50 = 1/2 = e^{28k}$$

$28k = \ln(1/2)$, $k = (1/28) \ln(1/2) = -\ln(2^{1/28})$ (result)

$y(t) = 50 \exp[-\ln(2^{1/28}) t]] = (50) 2^{-t/28}$ (result)

$y(40) = (50) 2^{-40/28} = (50) 2^{-10/7} \approx 19$ cells (result)