Basics of 2nd Order, Ordinary Differential Equations/Method of Reduction

In a Nut Shell: A differential equation, $\mathbf{y'' + p(x)y' + q(x)y = 0}$ is of the second order since the second derivative of the dependent variable, y, appears in the d.e. This differential equation. appears frequently in mechanical systems. (i.e. vibration applications)

The principle of superposition applies to second order, linear, ordinary d.e.'s.

The sum of any two linearly independent solutions, y_1 and y_2 of the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

$y = c_1 y_1 + c_2 y_2$ is again a solution of the differential equation where c_1 and c_2 are constants.

provided that y_1 and y_2 are linearly independent.

Again Note: The two functions, y_1 and y_2, **must be linearly independent.**

Two functions , y_1 and y_2, are **linearly independent** provided that neither is a constant multiple of the other. **The Wronskian, WR,** is nonzero if the functions y_1 and y_2 are linearly independent. (below det means a 2 by 2 determinant)

$$WR = \det \begin{vmatrix} y_1 & y_2 \\ d\,y_1/dx & d\,y_2/dx \end{vmatrix} \neq 0$$

NOTE: A second order d.e. requires two initial condition in order to evaluate the

two constants of integration when integrating the d.e.

$$y'' + p(x)\,y' + q(x)\,y = 0$$

The initial conditions may appear as $y(a) = c$, $y'(a) = d$, where c and d are constants

If the right hand side of the d.e. is zero, the d.e. is said to be **homogeneous.**

$$a\,y'' + by' + cy = 0 \qquad \text{where } a, b, \text{ and } c \text{ are constants}$$

If you assume an "exponential" such as $y(x) = e^{rx}$ and substitute it into this d.e., then the result (factoring out e^{rx}) is:

The characteristic equation is $ar^2 + br + c = 0$

This characteristic equation has four possible roots using the quadratic formula for this second order quadratic, algebraic equation. The solution using the appropriate roots is called the "complementary solution" of the differential equation. Note that the values of the roots depend on the values of the constants a, b, and c. See the table below

Possibilities for roots, r	Complementary solution, y_c
distinct real roots r_1 and r_2	$y_c = C_1 \exp(r_1 x) + C_2 \exp(r_2 x)$
repeated roots real r_1 and r_1	$y_c = C_1 \exp(r_1 x) + C_2 x \exp(r_1 x)$
distinct imaginary roots, ir_1 and $-ir_1$	$y_c = C_1 \sin r_1 x + C_2 \cos r_1 x$
complex roots $r_1 \pm i r_2$	$y_c = e^{r_1 x} (C_1 \sin r_2 x + C_2 \cos r_2 x)$
repeated complex roots $r_1 \pm i r_1$	$y_c = e^{r_1 x} (C_1 \sin r_1 x + C_2 x \cos r_1 x)$

In a Nut Shell: Roots of the characteristic equation $ar^2 + br + c = 0$ for a second order linear differential equation with constant coefficients include distinct real roots, distinct imaginary roots, distinct complex roots, repeated real roots, and repeated imaginary roots, and repeated complex roots. Recall the solutions or any linearly independent combination of solutions to a differential equation is also a solution.

Linear Combination of distinct real roots, a and - a .

The solutions in exponential form are: $y_1 = e^{ax}$ and $y_2 = e^{-ax}$

So the **complementary solution** can be written as $y_c = C_1 e^{ax} + C_2 e^{-ax}$

Now any linearly independent combination of these solutions is also a solution.

Take $y_3 = (y_1 + y_2)/2$ and $y_4 = (y_1 - y_2)/2$ or

$y_3 = (e^{ax} + e^{-ax}) /2 = \cosh (ax)$ and $y_4 = (e^{ax} - e^{-ax}) /2 = \sinh (ax)$

So $y_c = C_3 y_3 + C_4 y_4$

And the equivalent **complementary solution** can also be written as:

$$y_c = C_1 \cosh (ax) + C_2 \sinh(ax)$$

Linear Combination of distinct imaginary roots, i a and - i a .

The solutions in exponential form are: $y_1 = e^{iax}$ and $y_2 = e^{-iax}$

So the **complementary solution** can be written as $y_c = C_1 e^{iax} + C_2 e^{-i\,ax}$

Now any linearly independent combination of these solutions is also a solution.

Use: $y_1 = e^{iax} = \cos(ax) + i\sin(ax)$ and $y_2 = e^{-iax} = \cos(ax) - i\sin(ax)$

Take $y_3 = (y_1 + y_2)/2$ and $y_4 = (y_1 - y_2)/2i$ or

$y_3 = (e^{iax} + e^{-iax})/2 = \cos(ax)$ and $y_4 = (e^{ax} - e^{-ax})/2i = \sin(ax)$

So $y_c = C_3\, y_3 + C_4\, y_4$

And the equivalent **complementary solution** can also be written as
:

$$y_c = C_1 \cos(ax) + C_2 \sin(ax)$$

Example: Find the "complementary solution" for the following second order, ordinary,

homogeneous D.E.

$$d^2y/dx^2 + 6dy/dx + 13y = 0$$

You can write this d.e. in operator notation $(D^2 + 6D + 13)\,y = 0$

Strategy: Assume a complementary solution of the form: $y = Ae^{rx}$

So $d^2y/dx^2 = A\,r^2 e^{rx}$, $dy/dx = A\,r\,e^{rx}$, and $y = Ae^{rx}$

Substitute into the d.e. yields

$$Ae^{rx}(r^2 + 6r + 13) = 0$$

Since $Ae^{rx} \neq 0$, the characteristic equation for r becomes:

$$r^2 + 6r + 13 = 0$$

with roots $-3 \pm 2i$ (using the quadratic formula)

The complementary solution, y_c , is:

$$y_c(x) = F e^{(-3 + 2i)x} + G e^{(-3 - 2i)x}$$

where F and G are undetermined constants (need two initial conditions)

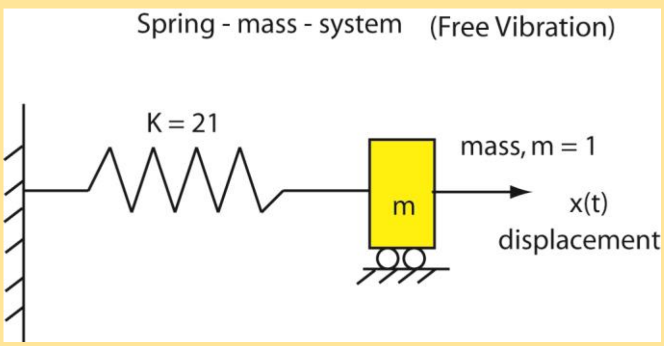
which can be expressed as follows, (**equivalent complementary solution**)

$$\boxed{y_c(x) = e^{-3x} (C_1 \sin 2x + C_2 \cos 2x)}$$

where C_1 and C_2 are again undetermined constants (determined by two initial conditions)

Example: Find the response, x(t) for the mechanical system (shown below) and described by the following differential equation.

$x'' + 21 x = 0$ with initial conditions $x(0) = 2$, $dx(0)/dt = 2$



Strategy: Assume $x(t) = C e^{rt}$ and put it into the differential equation to obtain the "characteristic equation".

$$C r^2 e^{rt} + 21 C e^{rt} = 0 \text{ or } C e^{rt} (r^2 + 21) = 0$$

$$r^2 + 21 = 0 \quad \text{giving} \quad r = \pm \sqrt{21}\, i$$

So the solution for x(t) becomes $x(t) = C_1 e^{\sqrt{21} i t} + C_2 e^{-\sqrt{21} i t}$

The equivalent solution for x(t) using sines and cosines is:

$$x(t) = C_1 \cos(\sqrt{21}\, t) + C_2 \sin(\sqrt{21}\, t)$$

where C_1 and C_2 are constants of integration determined from the initial conditions.

Also $dx(t)/dt = -\sqrt{21} C_1 \sin(\sqrt{21}\, t) + \sqrt{21} C_2 \cos(\sqrt{21}\, t)$

From $x(0) = 2$, $C_1 = 2$ and from $dx(0)/dt = 2$, $C_2 = 2 / \sqrt{21}$

and $$x(t) = 2 \cos(\sqrt{21}t) + (2/\sqrt{21}) \sin(\sqrt{21}t)$$

Method of Reduction for Second Order Differential Equations

In a Nut Shell: If one solution for a linear, second or higher order, differential equation is known, then it can be used to reduce the order of the differential equation leading to its general solution. This method applies to linear differential equations with either constant or variable coefficients such as:

Second order with constant coefficients: $ay'' + by' + cy = 0$

Third order with constant coefficients: $ay''' + by'' + cy' + dy = 0$

Second order with variable coefficients: $y'' + p(x)\, y' + q(x) = 0$

Third order with variable coefficients: $y''' + p(x)\, y'' + q(x)\, y' + r(x)\, y = 0$

Strategy: Suppose $y_1(x)$ is a solution to the original differential equation. Then introduce a new dependent variable, $v(x)$, and combine it with $y_1(x)$ to obtain a second solution, $y(x)$, for the original differential equation:

$$y(x) = y_1(x)\ v(x)$$

Procedure: Introduce this $y(x)$ into the differential equation to obtain a new one involving the new dependent variable, $v(x)$. As an **intermediate check on your procedure,** this substitution should result in a differential equation with the derivatives of $v(x)$, that with the introduction of a second dependent variable, say $w(x)$, can be integrated or separated leading to a differential equation of lower order. Once you arrive at this lower order differential equation in terms of derivatives of $w(x)$, you then integrate to find $v(x)$. Finally combine $v(x)$ with $y_1(x)$ to obtain the general solution of the original differential equation.

Note: The method of reduction provides you an alternative method to find general solutions of second or higher order, linear differential equations with constant or variable coefficients provided you are either given one solution or can find one.

Example: Find the general solution for the following second order, ordinary, homogeneous D.E. with variable coefficients using the method of reduction.

$$x^2\, d^2y/dx^2 + 3x\, dy/dx + y = 0 \qquad (1)$$

where $y_1(x) = 1/x$ is one solution of the differential equation.

Strategy: Assume the solution for $y(x) = (1/x)\, v(x)$ and substitute it into eq. (1).

First calculate derivatives: $y = (1/x)\, v$

$$dy/dx = -(1/x^2)\, v + (1/x)\, dv/dx$$

$$d^2y/dx^2 = (1/x)\, d^2v/dx^2 - (2/x^2) dv/dx + (2/x^3)\, v$$

Then substitute into eq. (1)

$$x^2\, d^2y/dx^2 = x\, d^2v/dx^2 - 2\, dv/dx + (2/x)\, v$$

$$3x\, dy/dx = 3\, dv/dx - (3/x)\, v$$

$$y = (1/x)\, v$$

and add (giving) $0 = x\, d^2v/dx^2 + dv/dx$ (2)

Next let $w(x) = dv/dx$ So (2) becomes $x\, dw/dx + w = 0$ (3)

Note: Eq (3) is a first order differential equation in terms of w(x), which is separable. Integrate (3) to obtain v(x).

$$dw/w = -dx/x \quad \text{giving} \quad \ln w = -\ln x + C_o \quad \text{or} \quad \ln (w\, x) = C_o$$

$$w\, x = C_2 \quad \text{or} \quad w = C_2/x = dv/dx \quad \text{so} \quad v(x) = C_2 \ln x + C_1$$

Finally: $y(x) = (1/x)\, v(x) = C_1 (1/x) + C_2 (1/x) \ln x$

which is the general solution of eq. (1) using the method of reduction. (result)

Example: Find the general solution for the following third order, ordinary,

homogeneous D.E. with constant coefficients using the method of reduction.

$$d^3y/dx^3 - d^2y/dx^2 - dy/dx + y = 0 \qquad (1)$$

Note: By inspection one of the solutions to eq. (1) is e^x .

Strategy: Assume the solution for $y(x) = e^x\, w(x)$ and substitute it into eq. (1).

First calculate derivatives: $y = e^x\, w$, $dy/dx = e^x\, w + e^x\, dw/dx$

$$d^2y/dx^2 = e^x\, w + 2\, e^x\, dw/dx + e^x\, d^2w/dx^2$$

$$d^3y/dx^3 = e^x\, w + 3\, e^x\, dw/dx + 3\, e^x\, d^2w/dx^2 + e^x\, d^3w/dx^3$$

Then substitute each derivative into eq. (1) and use a common factor of e^x to obtain:

$$e^x [w + 3 w' + 3 w'' + w'''] + e^x [- w - 2 w' - w''] +$$

$$e^x [- w - w'] + e^x [w] = 0 \qquad \text{Collect terms:}$$

The resulting d.e. simplifies to $w''' + 2 w'' = 0$ (2)

Next introduce a new dependent variable, $u(x) = w''$

So eq. (2) becomes a first order d.e.in the new dependent variable, $u(x)$:

$$du/dx + 2u = 0 \qquad (3)$$

Separate variables and integrate: $du/u = - 2 dx$, $\ln u = - 2x + C_1$

$u(x) = C_2 e^{- 2x} = w''(x)$ Now integrate w(x) twice to obtain w(x).

$dw/dx = C_2 e^{- 2x} + C_3$ and $w(x) = C_1 e^{- 2x} + C_2 x + C_3$

Recall the solution for y(x) is the product of $e^x w(x) = y(x)$

After multiplication: $y(x) = C_1 e^{- x} + C_2 x e^x + C_3 e^x$

which is the general solution of eq. (1) using the method of reduction. (result)

where $= C_1$, C_2, and C_3 are arbitrary constants of integration.