Particular Solutions of Nonhomogeneous Differential Equations

In a Nut Shell: If the right hand side of the d.e. is a specified function, f(x), then the second order, ordinary differential equation with constant coefficients is nonhomogeneous.

$a\, d^2y/dx^2 + b\, dy/dx + c\, y = f(x)$ (nonhomogeneous d.e.)

$a\, d^2y/dx^2 + b\, dy/dx + c\, y = 0$ (homogeneous d.e.)

where a, b, and c are constants and f(x) is a specified function

Strategy: The objective is to find functions that satisfy the differential equation and yield f(x). The first step in finding the particular solution is to set the right hand side of the differential equation to zero and proceed to find the solution of the homogeneous equation, $y_H(x)$.

Note: The functions appearing in the particular solution must differ (be linearly independent) from those in the complementary solution. You are then ready to move on to finding the particular solution.

More Background: There are two methods for finding particular solutions of nonhomogeneous d.e.'s. The method of undetermined coefficients and the method of variation of parameters. This section introduces the method of undetermined coefficients.

When can you use the method of undetermined coefficients? The answer is:

The method of undetermined coefficients can be used to find the particular solution if the function, f(x), on the right hand side of the d.e. is one of the following types:

■ A polynomial	■ Sine functions	■ Cosine functions
■ Sine and Cosine functions	■ Exponential functions	■ Sums or products of these functions (Not quotients)and ■

Reminder: You need to first find the complementary solution to the homogeneous equation so as not to repeat its functions in the particular solution,

In a Nut Shell: For functions, f(x), involving polynomials, trig functions, exponential functions, and **products** of these functions, you can find particular solutions, y_p , of nonhomogeneous, linear differential equations with constant coefficients**.** **Quotients** are not included.

$y'' + A\, y' + B\, y = f(x)$ **using the method of undetermined coefficients.**

Strategy: Find the terms of the particular solution by taking derivatives of f(x) up to the highest order of the d.e. For the second order d.e. illustrated above, you need to take two derivatives of f(x). Then the particular solution is a linear combination of terms from f(x), df(x)/dx, and $d^2f(x)/dx^2$. Use these functions unless they appear in the complementary solution.

For example if f(x) is 23 sin 2x, then f((x), f '(x), and f ''(x) involve sin 2x and cos 2x. The assumed form of the particular solution would be y_p = A sin 2x + B cos 2x .

If f(x) contains several terms, such as $f(x) = xe^x + 3x$, you may investigate the particular solution for each function individually to simplify the procedure. i.e. Let $f_1(x) = xe^x$ and $f_2(x) = 3x$. $f(x) = f_1(x) + f_2(x)$. Note again, $y_{1P}(x)$ and $y_{2P}(x)$ must be linearly independent from the terms in the homogeneous solution, $y_{1H}(x)$ and $y_{2H}(x)$.

The table below summarize simple cases for f(x) for which one can apply the method of undetermined coefficients to find the particular solution of nonhomogeneous d.e.'s.

Case 1: $f(x) = \text{polynomial} = a_1x^m + a_2 x^{m-1} + \ldots + a_n$

$$y_p(x) = A_mx^m + A_{m-1} x^{m-1} + \ldots + A_o$$

Case 2: $f(x) = a \cos kx + b \sin kx$

$$y_p(x) = A \cos kx + B \sin kx$$

Case 3: $f(x) = a e^{kx} + b e^{-kx}$

$$y_p(x) = A e^{kx} + B e^{-kx}$$

Case 4: Note - f(x) could involve products of the functions in Cases 1 – 3

NOTE: If any of the functions in the particular solution appear in the complementary solution, then you need to create new functions in your particular solution that are linearly independent of those in the homogeneous solution by multiplying them by an appropriate value of x such as x, x^2, x^3, etc. to obtain linearly independent functions.

Example: Find the particular solution for the d.e. $y'' - 4y = 2e^{2x}$

Strategy: First find the complementary solution to the homogenous d.e.

$$y'' - 4y = 0$$

So the characteristic equation is $r^2 - 4 = 0$, and roots $r = \pm 2$,

and the complementary solution is $y_c = C_1 e^{2x} + C_2 e^{-2x}$

For the nonhomogeneous d.e. $f(x) = 2e^{2x}$ Find the first two derivatives

$$df/dx = 4e^{2x} \qquad d^2f/dx^2 = 8e^{2x}$$

So the particular solution is $y_p = C_3 e^{2x}$ **provided** that it does not appear in the complementary solution to the homogeneous d.e. **But in this case, it does!**

Strategy: Multiply y_p by x to get a linearly independent function.

Therefore the particular solution takes the form: $y_p = C_3 \, x e^{2x}$

Strategy: Proceed with the method of undetermined coefficients by putting y_p into the d.e.

$$y'' - 4y = 2e^{2x}$$

Note $d^2 y_p/dx^2 = 4C_3 e^{2x} + 4C_3 \, x e^{2x}$

so $d^2 y_p/dx^2 - 4y_p = 4C_3 e^{2x} + 4C_3 \, x e^{2x} - 4C_3 \, x e^{2x} = 4C_3 e^{2x}$

Find the undetermined coefficient, C_3, by comparing coefficients on the left hand side of the d.e. with its right hand side as follows:

$4C_3 e^{2x} = 2e^{2x}$ and therefore $C_3 = ½$

So the particular solution becomes $y_p = ½ \, x e^{2x}$ (result)

Example: Consider a nonhomogeneous d.e. with a more complicated function, f(x), where none of its terms appear in the complementary solution, then find the particular solution for the following d.e.

$$y''' + 9y' = x \sin x + x^2 e^{2x}$$

Strategy: First find the complementary solution to the homogeneous d.e.

$$y''' + 9y' = 0$$

So the characteristic equation for the homogenous d.e. is $r^3 + 9r = 0$

and the three roots are as follows: $r = 0, \; r = \pm 3i$

The complementary solution is $y_c = C_1 + C_2 \cos 3x + C_3 \sin 3x$

For the nonhomogeneous d.e. $f(x) = x \sin x + x^2 e^{2x}$ Find the first three derivatives.

Note: f, f', f'' and f''' contain the following functions:

$\sin x$, $\cos x$, $x \cos x$, $x \sin x$, e^{2x}, $x e^{2x}$, $x^2 e^{2x}$

So the particular solution is a linear combination of these terms since none of these terms appears in the complementary solution.

To simplify, break $f(x)$ into two parts. Let $f_1(x) = x \sin x$ and $f_2(x) = x^2 e^{2x}$.

For $f_1(x)$: pick $y_{P1}(x) = (A + Bx) \cos x + (C + Dx) \sin x$

Calculate the derivatives, $y'_{P1}(x)$ and $y'''_{P1}(x)$, and substitute into the d.e.. using the method of undetermined coefficients.

The result is: $x \sin x = (-8A + 6D) \sin x + (8C + 6B) \cos x$

$$- (1/8) B x \sin x + 8 D x \cos x$$

Solve for A, B, C, and D by equating like terms on each side of the equal sign.

Equate like terms: $-8A + 6D = 0$, $8C + 6B = 0$, $-(1/8) B = 1$, $8D = 0$

which gives: $A = 0$, $B = -1/8$, $C = 3/32$, and $D = 0$.

The result is $y_{P1}(x) = -(1/8)\, x \cos x + 3/32 \sin x$.

Next calculate the particular solution for $f_2(x) = x^2 e^{2x}$.

$$y''' + 9y' = x \sin x + x^2 e^{2x}$$

The first three derivatives for $f_2(x) = x^2 e^{2x}$ include: $x^2 e^{2x}$, $x e^{2x}$, and e^{2x} .

So pick a linear combination for $y_{P2}(x)$ such as: $y_{P2}(x) = (A + Bx + Cx^2)\, e^{2x}$

Calculate the derivatives, $y'_{P2}(x)$ and $y'''_{P2}(x)$, and substitute into the d.e.

The result is: $x^2 e^{2x} = [(26A + 21B + 12C) + (26B + 42C)x + 26Cx^2]\, e^{2x}$

Equate like terms by the method of undetermined coefficients which gives:

$$26A + 21B + 12C = 0$$

$$26B + 42C = 0$$

$$26C = 1$$

The result is: $A = 570/(26^3)$, $B = -42/(26^2)$, and $C = 1/26$

The particular solution for $f_2(x) = x^2$ is then:

$$y_{P2}(x) = [\,(570/(26^3)) - (42/26^2)\,x + (1/26)\,x^2\,]\; e^{2x}$$

The result for the total particular solution is:

$$y_p(t) = -(1/8)\,x \cos x + 3/32 \sin x + [\,(570/(26^3)) - (42/26^2)\,x + (1/26)\,x^2\,]\; e^{2x}$$

Example: Find the particular solution for the d.e. $y''' + y'' = 3e^{x} + 4x^2$

Strategy: First find the complementary solution to the homogeneous d.e.

$$y''' + y'' = 0$$

The characteristic equation for the homogenous d.e. is $r^3 + r^2 = 0$

So the three roots are as follows: $r = 0$, 0, $r = -1$ **(r = 0 is a double root)**

The complementary solution is: $y_c = C_1 + C_2\,x + C_3\,e^{-x}$

For the nonhomogeneous d .e. $f(x) = 3e^{x} + 4x^2$ find the first three derivatives.

Note: f, f ‘, f ‘’, and f ‘’’ contain the following functions:

e^{x} , 8, x, and x^2 so initially take $y_p = C_4 e^x + C_5 x^2 + C_6 x + C_7$

for the trial function for the particular solution.

But in seeking linearly independent functions you must avoid duplicating those

functions appearing in the complementary solution, y_c i.e. $C_1 + C_2\,x$

Strategy: Multiply the terms in the particular solution C_5x^2, $C_6\,x$, and C_7 by x^2 .

So the modified particular solution becomes:

$$y_p = C_4\,e^{x} + C_5\,x^4 + C_6\,x^3 + C_7\,x^2$$

Now use the method of undetermined coefficients. Calculate the derivatives y_p'' and y_p'''.

$y_p'' = C_4 e^x + 12 C_5 x^2 + 6 C_6 x + 2 C_7$

$y_p''' = C_4 e^x + 24 C_5 x^2 + 6 C_6 x$

So $y_p''' + y_p'' = 24 C_4 e^x + 12 C_5 x^2 + (24 C_5 + 6 C_6) x + 6 C_6 + 2 C_7$

Which must equal the right hand side $= 3e^x + 4x^2$

Compare coefficients. The result is $C_4 = 3/2$, $C_5 = 1/3$, $C_6 = -4/3$, $C_7 = 4$

So the particular solution $y_p = 3/2\, e^x + 1/3\, x^4 - 4/3\, x^3 + 4x^2$ (result)