Integrals Using Substitution/Substitution and Integration by Parts

In a Nut Shell: Some integrals may be simplified using a simple substitution. In others you may need to combine one or more substitutions along with one or more integration by parts (in either order). Once again the objective is to transform the original integral into one or more standard integrals.

Below is an example using a simple substitution. i.e. $w = 2x^3 + 5$

Example: Given: $\int_7^{15} f(x)\, dx = 23$ and $\int_{21}^{15} (f(x)\, dx = 13$

Using these integrals and the substitution suggested above to **evaluate I** where

$$I = \int_1^2 48\, x^2\, f(2x^3 + 5)\, dx$$

Use $w = 2x^3 + 5$ then

$dw = 6x^2\, dx$ Therefore $48\, x^2\, dx = 8\, dw$ If $x = 1$, $w = 7$ and if $x = 2$, $w = 21$

$$I = \int_7^{21} 8\, f(w)\, dw = 8 \int_7^{21} f(w)\, dw$$

From above $\int_7^{21} f(x)\, dx = \int_7^{15} f(x)\, dx + \int_{15}^{21} f(x)\, dx = 23 - 13 = 10$

As a result $I = (8)\,(10) = 80$ (final result)

Here's an example **Combining a Substitution with Integration by Parts**

i.e. Substitution and $I = \int u\, dv = uv - \int v\, du$

Example: $I = \int (x^3 / \sqrt{(1 - x^2)})dx$

Substitution: First substitution is: $x = \sin\theta$, $dx = \cos\theta\, d\theta$

now $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$

So $I = \int (\sin^3\theta \cos\theta / \cos\theta)\, d\theta = \int \sin^3\theta\, d\theta$

Since the integrand is cubic, try rewriting as follows:

Write as $\int \sin^2\theta \sin\theta\, d\theta$

and **now integrate by parts**

Integration by Parts

let $u = \sin^2 \theta$, $\qquad dv = \sin \theta \, d\theta \leftarrow$ Standard integral

$\qquad du = 2 \sin \theta \cos \theta \, d\theta \qquad v = -\cos \theta$

Then I becomes $-\sin^2 \theta \cos \theta + 2\int \sin \theta \cos^2 \theta \, d\theta$

Now let $w = \cos \theta$, $dw = -\sin \theta \, d\theta$ and integral becomes
one in standard form:

$I = -\sin^2 \theta \cos \theta - 2\int w^2 \, dw = -\sin^2 \theta \cos \theta - (2/3) \, w^3 + c$

$\qquad$ Or $I = -\sin^2 \theta \cos \theta - (2/3)\cos^3 \theta + c$

Finally, express result in terms of original independent variable x

Recall $\sin \theta = x$ then $\cos \theta = \sqrt{(1 - x^2)}$ so

Result: $I = -x^2 \sqrt{(1 - x^2)} - (2/3)(1 - x^2)^{3/2} + C$