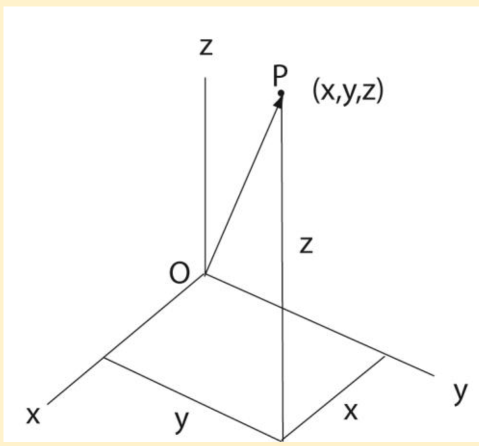
Uses of Rectangular, Cylindrical and Spherical Coordinates

In a Nut Shell: Three options are available in the evaluation of double and triple integrals. The table below lists these options.

- Switching from one set of coordinates to another
- Changing the order of integration
- Changing the variables of integration

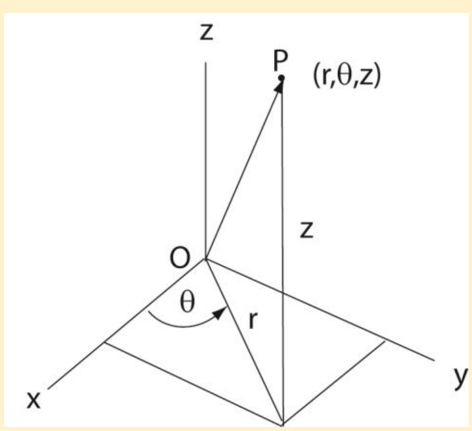
This section focuses on uses of rectangular, cylindrical, and spherical coordinates in the evaluation of double and triple integrals. (The first option)

Rectangular Coordinates of a point, P, in space are: (x, y, z) as shown below



Cylindrical Coordinates of a point, P, in space are: (r , θ, z)

where θ = angle between the x-axis and the radius, r, in the x-y plane as shown below



So the rectangular coordinates (x, y, z) of P in cylindrical coordinates are:

$x = r \cos \theta, \quad y = r \sin \theta, \quad z = a$

Spherical Coordinates of a point ,P, in space are: (ρ, θ, φ)

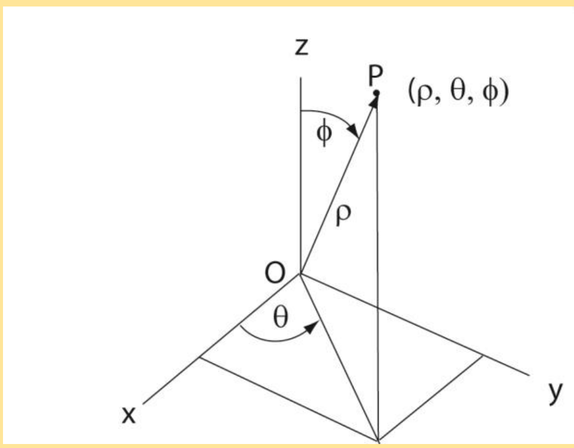
Let ρ = magnitude of vector from the origin, O, out to the point P

θ = angle between the x-axis and the line formed by the projection of ρ

on to the x-y plane NOTE: This projection is the same as r in cylindrical

coordinates so θ has the same meaning for both spherical and cylindrical coordinates

φ = angle between the z-axis and ρ



So the rectangular coordinates (x, y, z) of P in spherical coordinates are:

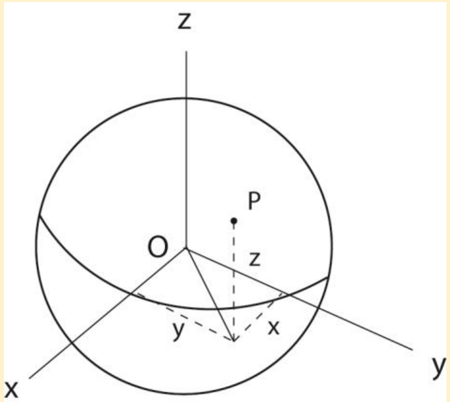
$$x = \rho \sin\varphi \cos\theta\,,\quad y = \rho \sin\varphi \sin\theta\,,\quad z = \rho\cos\varphi$$

An example that illustrates uses of rectangular, cylindrical and spherical coordinates systems in evaluating an integral is calculating the volume of a sphere. **Note** which of these options simplifies the determination of the limits of integration and/or evaluation of the integral.

Volume of a Sphere using Rectangular, Cylindrical, and Spherical Coordinates

Example: Find the volume, V, of a sphere of radius, R, first using rectangular coordinates.

Use a Type 1 region where the projection is on the x y - plane and integrate in the z-direction first. See the figure below.



The projection of the sphere on to the xy-plane is a circle of radius R. Use symmetry by evaluating 1/8th of the volume and multiplying by 8 to get the total volume. Thus the integral becomes

$$V = 8 \int_{x=0}^{x=R} \int_{y=0}^{y=\sqrt{(R^2 - x^2)}} \int_{z=0}^{z=\sqrt{(R^2 - x^2 - y^2)}} [\ dz\ ]\ dy\, dx$$

The first integration gives

$$V = 8 \int_{x=0}^{x=R} \int_{y=0}^{y=\sqrt{(R^2 - x^2)}} \sqrt{(R^2 - x^2 - y^2)}\ dy\, dx$$

Notice that the integral on the variable y has the form $(a^2 - y^2)$ where $a^2 = R^2 - x^2$. So you could proceed by using a substitution $y = a \sin \alpha$. But at this point you might instead switch to “polar coordinates” for the double integral. Then the integral becomes

$$V = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=R} \sqrt{(R^2 - r^2)}\ r\, dr\, d\theta$$

Let $w = R^2 - r^2$ so $dw = - 2r\, dr$ and $r\, dr = - ½\, dw$

The second integration becomes

$$V = 8 \int_{\theta=0}^{\theta=\pi/2} \int_{w=R^2}^{w=0} w^{1/2}\ (- ½)\ dw\, d\theta$$

$$V = - 4 \int_{\theta=0}^{\theta=\pi/2} (2/3)\, w^{3/2} \Big|_{R^2}^{0}\ d\theta$$

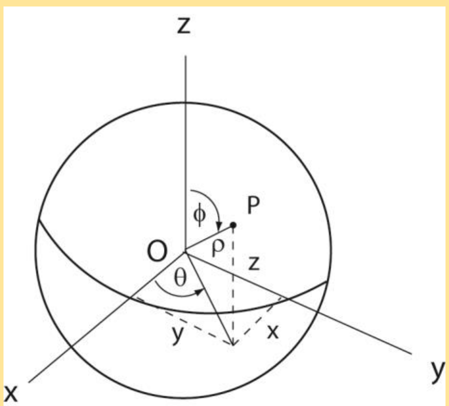
Finally integrate on the variable θ

$$V = -4 \int_{\theta=0}^{\theta=\pi/2} (-2/3)\, R^3 \, d\theta$$

which gives $V = 4\pi R^3 / 3$ (result)

Next find the volume of the sphere using spherical coordinates.

You will see that the use of spherical coordinates simplifies determining the limits of integration.



First integration on the variable ρ.

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \int_{\rho=0}^{\rho=R} \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} \rho^3/3\Big|_0^R \sin\varphi \; d\varphi\, d\theta$$

Second integration on the variable φ .

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{\varphi=0}^{\varphi=\pi} (R^3/3) \sin\varphi \; d\varphi\, d\theta = (R^3/3) \int_{\theta=0}^{\theta=2\pi} -\cos\varphi\Big|_0^\pi \, d\theta$$

Third integration on the variable θ .

$$V = \int_{\theta = 0}^{\theta = 2\pi} 2 \ R^3 / 3 \ \ d\theta = 4 \pi R^3 / 3 \qquad \text{(result)}$$

Note the ease of determining the limits of integration when using spherical coordinates when compared to the use of rectangular coordinates to evaluate the integral.