Local Maxima and Minima of Functions with Two Variables

In a Nut Shell: Maximum and minimum values of functions in the case of two independent variables, (x,y), are similar to the case with one independent variable, x, in that they occur where the slope at the "critical location" i.e. (a,b) is zero.

A **local maximum** occurs near (a,b) if $f(x,y) \le f(a,b)$. Likewise a **local minimum** occurs near (a,b) if $f(x,y) \ge f(a,b)$. An **absolute maximum** occurs at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in the domain of the function and an **absolute minimum** occurs at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in the domain of the function.

It is possible that the function, f(x,y), has neither a maximum nor a minimum at a critical point, (a,b). In this case point (a,b) is called a "saddle point".

Locating critical points of a function, f(x,y):

The slope of the function, f(x,y), must be zero at each critical point, (a,b). For functions of two independent variables, (x,y), the following partial derivatives must hold:

$$\partial f(a,b)/\partial x = 0 \quad \text{and} \quad \partial f(a,b)/\partial y = 0$$

Procedure for finding Local Minimum, Local Maximum, and Saddle Points of f(x,y)

Locate critical points, (a,b)	$\partial f(a,b)/\partial x = 0$ and $\partial f(a,b)/\partial y = 0$
Calculate D(a,b)	$D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [\, f_{xy}(a,b)]^2$
For a local minimum	$D > 0$ and $f_{xx}(a,b) > 0$
For a local maximum	$D > 0$ and $f_{xx}(a,b) < 0$
For a saddle point	$D < 0$
Test Fails	If $D = 0$

Example: Locate all critical points and identify local maxima, local minima, and any saddle points for the function

$$f(x,y) = (1/3)\,x^3 + xy^2 - 2xy$$

Locate critical points, (a,b)	$\partial f/\partial x = x^2 + y^2 - 2y = 0$ (equation 1) and $\partial f/\partial y = 2xy - 2x = 0$ $2x(y - 1) = 0$ (equation 2)
Use equation 2. Case 1: $x = 0$ Put into equation 1.	$y(y - 2) = 0$ Therefore: $y = 0$ and $y = 2$ Critical point are: (0,0) and (0,2)
Use equation 2. Case 2: $y = 1$ Put into equation 1.	$x^2 - 1 = 0$ So $x = \pm 1$ Critical points are: (1,1) and (- 1,1)
Calculate D(a,b)	$f_{xx} = 2x$, $f_{yy} = 2x$, $f_{xy} = 2y - 2$ and $D = f_{xx}f_{yy} - f_{xy}^2$

Strategy:

Set up table to identify local max, local min, and saddle points for f(x,y)

Results:

x,y	f_{xx}	f_{xy}	f_{yy}	D	Type
0,0	0	- 2	0	-	saddle
0,2	0	2	0	-	saddle
1,1	2	0	2	+	local min
-1,1	- 2	0	-2	+	local max