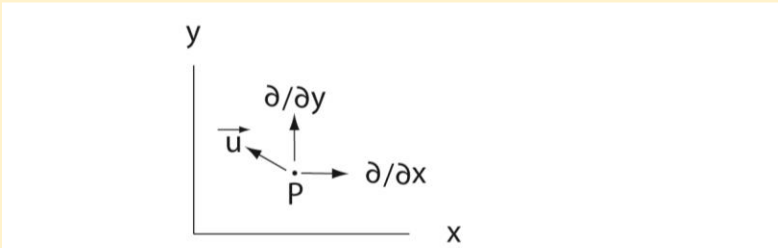
The Directional Derivative, its Definition and Physical Interpretation

In Nut Shell: The simple derivative of a function, y, of one independent variable, x, can be viewed as giving the derivative of y in the direction of x. This same concept extends to functions of more than one independent variable and any given direction.

The directional derivative of a function, $D_u f(x,y)$, gives the rate of change of f(x,y) on a line through the point, P, in the direction of the unit vector, **u**. See the figure below.



∂/∂x represents the change of the function (slope) in the x-direction,

∂/∂y represents the change of the function (slope) in the y-direction,

and D_u f represents the change of the function (slope) through P in the direction of the unit vector **u**.

One can show that the directional derivative of f(x,y) in the direction of the unit vector, **u,** D_u f(x,y) , is the dot product of the gradient of f(x,y) with the unit vector, **u**. It provides a convenient method to calculate the directional derivative.

$$D_u\, f(x,y) = \text{grad } f(x,y) \cdot \mathbf{u}$$

Recall that the gradient function, grad f(x,y), points in the direction in which the function f(x,y) increases (or decreases) most rapidly and the dot product with **u** gives the component of grad f(x,y) in the direction of **u .**

The maximum value of the directional derivative is in the direction of grad f(x,y). So it occurs when **u** is a unit vector in the direction of grad f .

So $\mathbf{u} = \text{grad } f \,/\, |\text{grad } f|$

i.e. If f (**x**) represents the temperature, then proceeding in the direction of **u** gives the greatest temperature change.

Suppose z = f(x,y) represents a surface in xyz. Further suppose that the surface

F(x,y,z) = z – f(x,y) is continuously differentiable. Then grad F is a vector normal

to the surface. Call this normal vector **n** .

$$\mathbf{n} = \partial F/\partial x\ \mathbf{i} + \partial F/\partial y\ \mathbf{j} + \partial F/\partial z\ \mathbf{k} = \text{grad } F$$

Here **n** is normal to the tangent plane at each point of the surface F(x, y, z)

So grad F can be useful to find tangent planes to surfaces and tangent lines.

Physical Interpretation of Directional Derivative

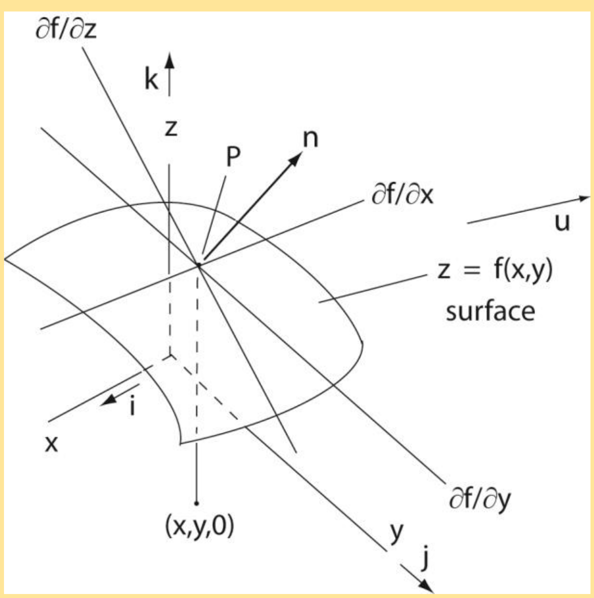
In the figure below:

$\partial f/\partial x$ represents the slope of the “surface” f(x,y) inthe x-direction at point P

$\partial f/\partial y$ represents the slope of the “surface” f(x,y) inthe y-direction at point P

$\partial f/\partial z$ represents the slope of the “surface” f(x,y) inthe z-direction at point P.

If **u** is any arbitrary unit vector, then the directional derivative of the function in the direction of **u** , D_u f(x,y), represents the slope of the function in the direction of **u** and can be calculated by the dot product , grad F · **u** = **n** · **u** .



Example Find the directional derivative of F(x,y), where F(x,y) is defined by $F(x, y) = 2x^2 + 3xy + 4y^2$ at the point, P(1,1) in the direction of the unit vector

$$\mathbf{u} = (\mathbf{i} + \mathbf{j})/\sqrt{2}$$

Strategy: Calculate the gradient of F and take the dot product with **u** to determine the directional derivative.

$$\text{grad } F = \partial F/\partial x \ \mathbf{i} + \partial F/\partial y \ \mathbf{j} = (4x + 3y)\,\mathbf{i} + (3x + 8y)\,\mathbf{j}$$

At point P (1,1) grad F = 7 **i** + 11 **j**

So $dF / d\mathbf{u} = [\,7\,\mathbf{i} + 11\,\mathbf{j}\,] \cdot [(\mathbf{i} + \mathbf{j})/\sqrt{2}\,] = 18/\sqrt{2}$ (result)

Example: From the previous example, $F(x,y) = T(x,y) = 2x^2 + 3xy + 4y^2$.
Suppose the function, T(x,y), represents temperature distribution.

Now **find the maximum change of the temperature at point P(1, 1).**

The maximum change will be given by the maximum directional derivative at point P and will be in the direction of grad T.

Strategy: Calculate the unit vector in the direction of grad T and then take the dot product with grad T.

$$\text{grad } T = (4x + 3y)\,\mathbf{i} + (3x + 8y)\,\mathbf{j}$$

At P(1, 1) (from above) grad T = 7 **i** + 11 **j**

So $\mathbf{u} = [\,7\,\mathbf{i} + 11\,\mathbf{j}\,] / \sqrt{(7^2 + 11^2)} = [\,7\,\mathbf{i} + 11\,\mathbf{j}\,] / \sqrt{(170)}$

So the maximum directional derivative = grad T · **u**

$$= [7\,\mathbf{i} + 11\,\mathbf{j}\,] \cdot [7\,\mathbf{i} + 11\,\mathbf{j}\,] / \sqrt{(170)}]$$

Maximum change in temperature at P $= (49 + 121) / \sqrt{(170)} = \sqrt{(170)}$ (result)