Existence and Uniqueness of Solutions of Linear Second Order Differential Equations

In a Nut Shell: Under restrictions solutions exist for linear second order differential equations that are unique throughout an interval. The theorem below details these restrictions.

Theorem for a general second order, ordinary differential equation of the form:

$$y'' + p(x)\, y' + q(x)\, y = g(x)$$

with initial conditions $y(x_o) = y_o,\ y'(x_o) = y'_o$

where $p(x)$, $q(x)$, and $g(x)$ are continuous functions on an open interval I that contains the point x_o .

Then there is only one solution , y, that exists throughout the interval I. i.e.

A solution exists for the differential equation .
The differential equation has only one solution. The solution is unique.
The solution exists throughout the interval I where the coefficients are continuous and twice differentiable in the interval.

Example: Determine the longest interval, I, for the following initial value application.

$$(x - 5)\, y'' + x\, y' + (\ln x)\, y = 0 \qquad y(3) = 0,\ y'(3) = 2$$

$$y'' + (x/(x - 5)\, y' + \ln x\ /\ (x - 5)\, y = 0$$

$x/(x - 5)$ continuous except for $x = 5$

$\ln x\ /\ (x - 5)$ continuous except for $x = 5$ with restriction $x > 0$

So the longest interval containing $x = 3$ is $(0, 5)$ (result)

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Example: Determine the longest interval, I, for the following differential equation.

$$(x^2 - 36)\ y^{(4)} + 6x^2 y''' + 18y = 0$$

$$y^{(4)} + [\,6x^2 / (x^2 - 36)\,]\,y''' + [\,18 / (x^2 - 36)\,]\ y = 0$$

$$y^{(4)} + [\,6x^2 / (x - 6)(x + 6)]\,y''' + [\,18 / (x - 6)(x+6)\,]\ y = 0$$

Points of discontinuity are at $x = 6$ and at $x = -6$

So the longest interval contains $(-\infty, -6)$ $(-6, 6)$ $(6, \infty)$ (result)