Changing Order of Integration for Multiple Integrals

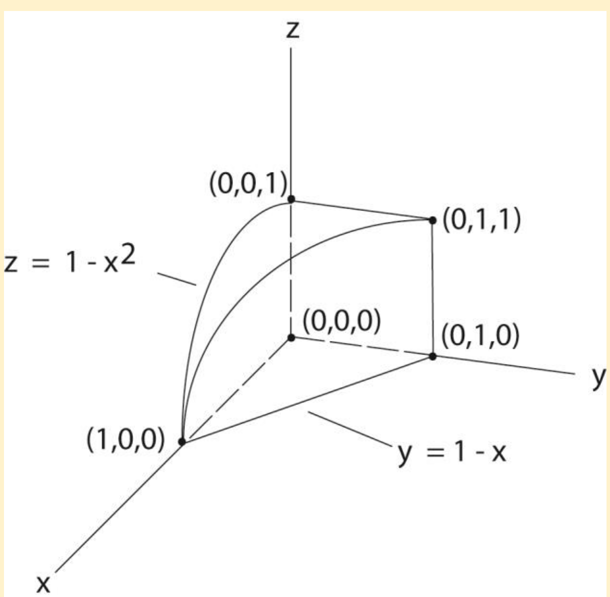
In a Nut Shell: The evaluation of double or triple integrals over a region, R, may be complicated due to the complexity of the integrand, due to the order of integration or both. The strategy is the same for both double and triple integrals but the triple integrals are generally harder since you are dealing with intersecting surfaces that are harder to visualize rather than intersecting curves.

Strategy: Identify the surfaces (or curves) that bound the region of integration by drawing the projections of the surfaces on various planes (or the curves). In so doing you will identify the limits of integration in each plane or the points of intersection of the curves. Then pick an order of integration that simplifies evaluation of the integral.

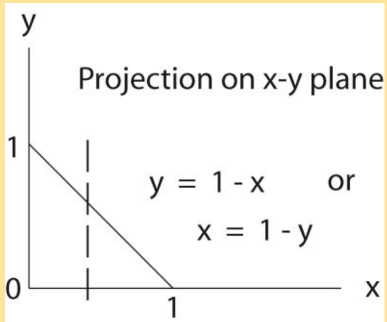
Example: Change the order of integration for the integral below from dy dz dx to dy dx dz.

$$I = \int_{x=0}^{x=1} \int_{z=0}^{z=1-x^2} \int_{y=0}^{y=1-x} f(x,y,z)\ dy\ dz\ dx$$

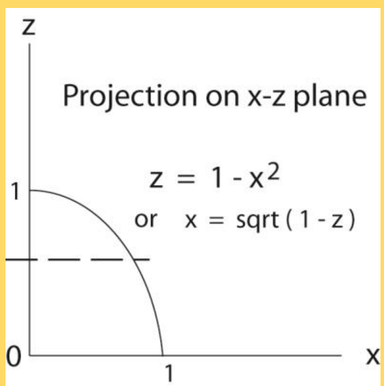
The first step is to visualize, R, the region of integration and the intersecting surfaces by using limits of integration to plot the region. i.e. $y = 1 - x$ is a plane in the z-direction. $z = 1 - x^2$ is a surface in the y-direction. See the figure below.



The next step is to draw the projections of the intersecting surfaces on to the xy and xz planes as shown below to obtain the limits of integration.



So $0 \le y \le 1 - x$ for the first integration on the variable, y



So $0 \le x \le \sqrt{(1 - z)}$ for the second integration on the variable, x

The remaining integration is on the third variable, z. In this case $0 \le z \le 1$.

$$I = \int_{z=0}^{z=1} \int_{x=0}^{x=\sqrt{(1-z)}} \int_{y=0}^{y=1-x} f(x,y,z)\, dy\, dx\, dz \qquad \text{(result)}$$

Two Dimensional Example:

Reverse the order of integration for the integral shown below.

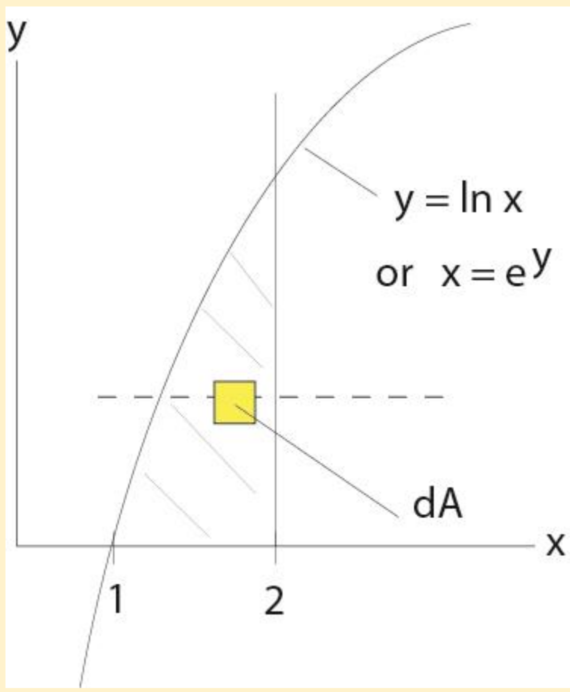
$$I = \int_{1}^{2} \int_{0}^{\ln x} f(x,y)\, dy\, dx$$

Strategy:

- Draw the figure using the limits of integration. (See the figure below.)
- Identify the intersection of the curves first in x and then in y directions.
- Use these points of intersection to establish the limits of integration in reverse order.

i.e. **"sweep"** element of area, dA, first in the x-direction then in the y-direction.

The limits of integration are $y = 0$ to $y = \ln x$ and $x = 1$ to $x = 2$.



Integration in the x-direction: $e^y \le x \le 2$

Integration in the y-direction: $0 \le y \le \ln 2$

So $$I = \int_{y=0}^{y=\ln 2} \int_{x=e^y}^{x=2} f(x,y)\, dx\, dy$$ (result)