Comparison Test for Improper Integrals

In a Nut Shell: Sometimes improper integrals involve complicated expressions that cannot be integrated. Yet, one still wants to determine if the improper integral converges or not. The "comparison test" provides a way to evaluate such integrals.

Strategy: Consider the original improper integral:

$$I = \int_a^\infty f(x)\,dx$$

where f(x) is a complicated function. The strategy is to find another integral with a simpler function, g(x), that you can evaluate

$$I = \int_a^\infty g(x)\,dx$$

where $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$.

Then **the comparison theorem provides** the following:

a. If $\int_a^\infty f(x)\,dx$ converges then so does $I = \int_a^\infty g(x)\,dx$

b. If $\int_a^\infty g(x)\,dx$ diverges then so does $I = \int_a^\infty f(x)\,dx$

Reasoning: If the area under the larger function, f(x), is finite, then the area under the smaller function, g(x), must also be finite. (converges) Likewise, if the area under the smaller function, g(x), is infinite, then the area under the larger function, f(x), must also be infinite. (diverges).

Strategy: Examine f(x) and reason whether it might converge or diverge.
Then pick g(x) appropriately.

Note: The actual value of the improper integral, if it converges, is not determined.

In a Nut Shell: The improper integral may involve several types of functions, products of functions, and/or quotients of functions that increase at different rates as the independent variable, x, increases towards infinity. Useful inequalities of several functions are given below.

Growth of functions as x increases without bounds :

$$\ln x \ll x^p \leq b^x \ll x^x$$

where p is a positive exponent

Example $I = \int_0^\infty ([x / (x^3 + 1)]dx$

Note that the denominator dominates for large values of x. So it **appears that the improper integral probably converges.**

Here $g(x) = x / (x^3 + 1)$. **So pick an f(x) that is larger than g(x).** Then if

$$\int f(x)\, dx \text{ converges so will } \int g(x)\, dx .$$

$$I = \int_0^\infty [x / (x^3 + 1)]dx = \int_0^1 ([x / (x^3 + 1)]dx + \int_1^\infty ([x / (x^3 + 1)]dx$$

Note the first integral is definite and the second one is improper.

Pick $f(x) = 1 / x^2$ and $g(x) = x / (x^3 + 1)$.

Note: $1 / x^2 > x (x^3 + 1)$ for large values of x. So $f(x) \geq g(x)$.

$$I_c = \int_1^\infty 1/ x^2 \, dx$$

$$I_c = \lim_{t \to \infty} (- 1/x) \Big|_1^t = 1 \quad \text{result: integral converges}$$

and since $f(x) \geq g(x)$ for large x, the original integral converges.

Example $I = \int_0^\infty ([1 / (x - e^{-x})]dx$ here $f(x) = 1 / (x - e^{-x})]$

Will this original integral converge or diverge? Note that the term e^{-x} dominates in the denominator. **So it appears that the integral probably diverges**. i.e.

Pick $g(x) = 1/x$ Note that $f(x) = 1 / (x - e^{-x}) > 1/x$ so $f(x) > g(x)$

$$I = \int_1^\infty ([1 /x]dx = \lim_{t \to \infty} \ln x \Big|_1^t = \infty - \ln 1 = \infty$$

Since $\int g(x)$ diverges (smaller area), the larger area $\int f(x)\, dx$ will also diverge