Slope Fields, Solution Curves, and Semi-stable Equilibrium Solutions

In a Nut Shell: First order differential equations of the form

$$dy/dx = f(x,y)$$

don't always yield solutions for y(x) since the dependent variable, y, appears on both sides of the equal sign. This d.e. may or may not be separable depending on the expression, f(x,y).

However, the left side, dy/dx, represents the slope of the function, f(x, y). Thus a table of values of x and y can be constructed to yield the slope of y(x) at each point (x,y). Connecting nearby slopes with a "smooth" curve then produces a "solution curve" in the **"slope field"** based on the starting point, (x_o,y_o)**). A family of curves yields a qualitative interpretation of the solution for y(x) for various starting points, (x_o, y_o)**

A separate class of differential equations, called autonomous, are of the form:

$$dy/dx = f(y)$$

Here the slope and the solution curves (a smooth curve formed by connection of the slopes)

are independent of x. So a family of curves translated in x are also solution curves.

Strategy for construction of the Slope Field:

The slope field consists of plots of the slopes at individual points.

Step 1	Set up a table of x and y values with appropriate increments.
Step 2	Calculate f(x,y) for each x and y value.
Step 3	Plot the slopes.
Step 4	Connect slopes with smooth curves, the **solution curves** for the d.e.
Step 5	If the differential equation is autonomous, then the type of stability may be of interest. In this case plot a phase diagram to determine if the critical points are stable, unstable, or semi-stable. Then proceed to plot the slope field and solution curves.

Example: Find the critical points then sketch the phase diagram for the following differential equation:

$$dy/dt = (1/2)\, y\, (y - 2)^2 (y - 4)$$

Strategy: Step 1: Find the critical points by setting $f(y) = 0$.

Critical Points: $f(y) = 0$ at $y = 0$, $y = 2$, and $y = 4$

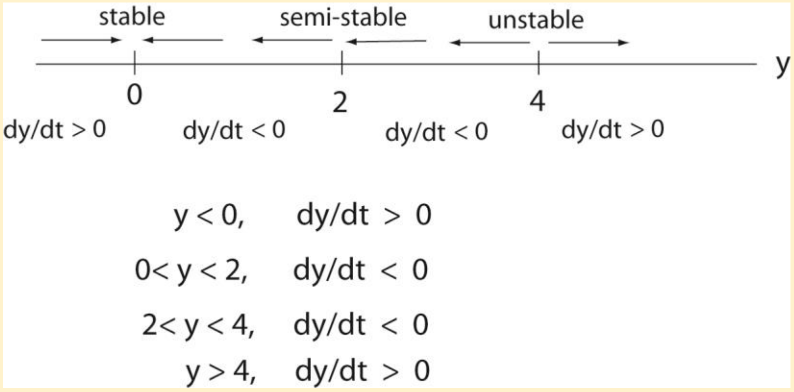
Step 2: Identify each region where slopes may change from positive to negative, stay the same, or change from negative to positive. Evaluate the slopes on each side of the critical points by taking values with each region.

The four regions in this example include: $y < 0$, $0 < y < 2$, $2 < y < 4$, $y > 4$

i.e. For $y = -1$, $dy/dt = 45/2 > 0$, for $y = 1$, $dy/dt = -3/2 < 0$

For $y = 3$, $dy/dt = -3/2 < 0$, and for $y = 5$, $dy/dt = 45/2 > 0$

Step 3: Plot Phase Diagram:



Note: A right directed arrow indicates that the dependent variable, y, is increasing.

A left directed arrow indicates that the dependent variable, y, is decreasing.

Results: At the critical point $y = 0$, the response is stable.

At the critical point $y = 2$, the response is semi-stable.

At the critical point $y = 4$, the response is unstable.

i.e. In the neighborhood of $y = 0$, the value of $y(t)$ tends toward a stable value.

In the neighborhood of $y = 2$, the value of $y(t)$ hovers around a stable value.

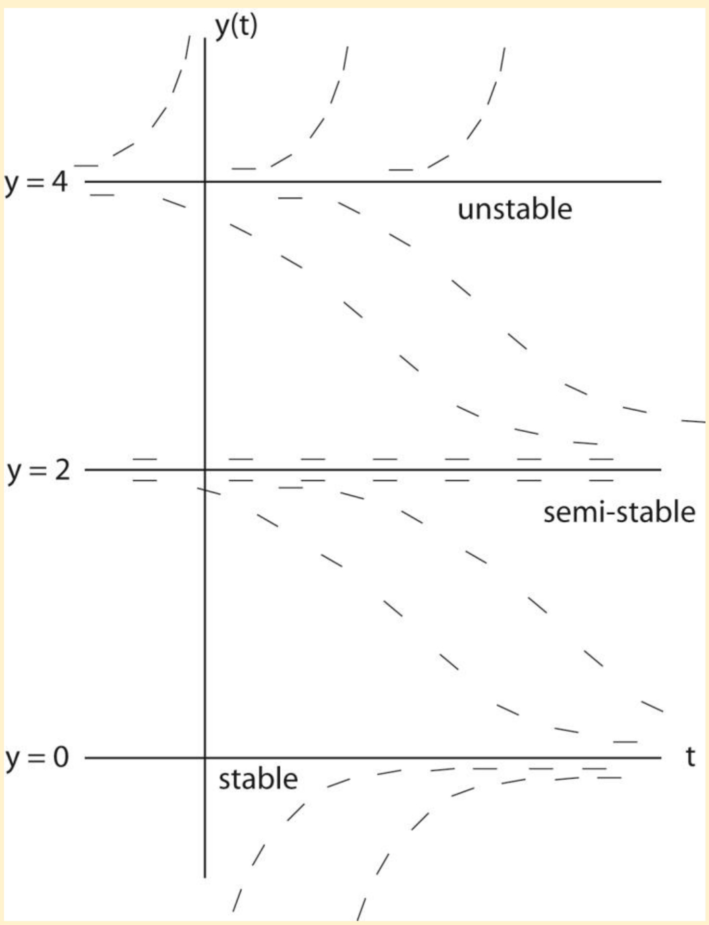
In the neighborhood of $y = 4$, the value of $y(t)$ departs from a stable value.

Example: Plot the slope field for: $dy/dt = (1/2)\, y\,(y - 2)^2\,(y - 4) = f(y)$

Strategy: Set up a table including values of y in each region and calculate the slopes, f(y).

Note: Values of slope are independent on t. So **"solution curves"** are the same for translations in t.

For $y > 4$	f(y) for 4.25, 4.5, 4.75, 5.0, etc	calculate slope and plot
For $2 < y < 4$	f(y) for 2.25, 2.5, 3.0, 3.5, 3.75, etc	calculate slope and plot
For $0 < y < 2$	f(y) for 1.9, 1.75, 1.5, 1.0, 0.5, etc	calculate slope and plot
For $y < 0$	f(y) for - 3.0, - 2.5, - 2.0, - 1.5, - 0.5 etc	calculate slope and plot



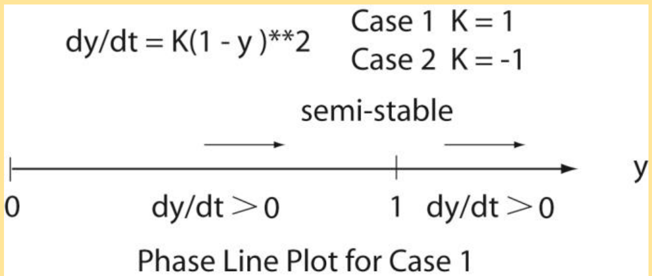
Plot of Slope Field Showing Solution Curves

Qualitative response for y(t) showing solution curves for stable, semi-stable, and unstable regions.

Autonomous Differential Equations - Semi-stable Equilibrium Solutions

In a Nut Shell: Semi-stable equilibrium solutions occur when the slope (whether positive or negative) remain the same across a critical point. The example below illustrates this for the autonomous differential equation, $dy/dt = K(1 - y)^2$.

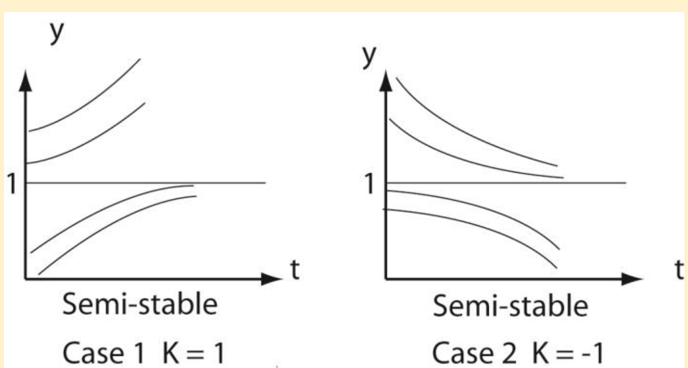
Example: Plot the slope field for: $dy/dt = K(1 - y)^2 = f(y)$
after drawing the phase plot. It will aid in the construction of the slope field.
The phase plot for f(y) is as follows:



Strategy: Set up a table including values of y in each region and calculate the slopes, f(y).

y	Case 1 K =1 dy/dt		Case 2 K = -1 dy/dt
0.000	1.000		- 1.000
0.250	0.5625		- 0.5625
0.500	0.250		- 0.250
0.750	0.0625		- 0.0625
1.000	0.000		0.000
1.250	0.0625		- 0.0625
1.500	0.025		- 0.025
1.750	0.5625		- 0.5625
2.000	1.000		- 1.000

Note: The values of slope do not depend on the time, t. The equilibrium solution is at y = 0. Use this horizontal line and the values of slope to construct the slope field.



Plot of Slope Field Showing Solution Curves