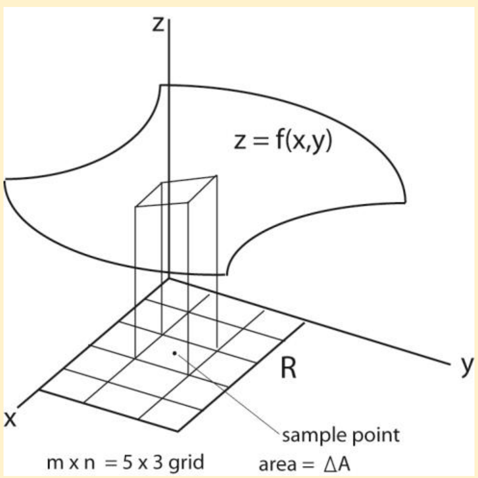
Approximate Volumes under a Surface using the Riemann Sum

In a Nut Shell: Premise The volume of a solid S that lies underneath the surface, f(x,y), and above the rectangle, R, in the xy-plane can be approximated by the Riemann Double Sum.

Volume $= \iint_R f(x,y)\, dA$ is replaced by the double sum as follows:

$$\iint_R f(x,y)\, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$$

where (x_{ij}^*, y_{ij}^*) is the sample point within each rectangular area in the xy-plane, $f(x_{ij}^*, y_{ij}^*)$ is the value of the surface at each sample point, and ΔA is the area of each rectangle in the xy- plane. The location of the sample points will impact the value of the volume determined. One possible sample point is the mid-point of each rectangular area in the xy-plane. But other sample points may be taken as well. See the figure below.



The finite sum of each individual "skyscraper" with a rectangular base area, ΔA, and height,

$f(x_{ij}^*, y_{ij}^*)$, yields an approximate value of the volume for each individual "skyscraper".

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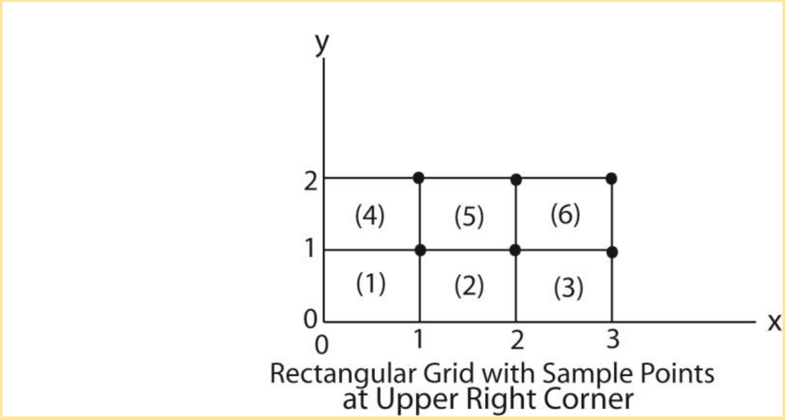
Example: Estimate the volume of a solid S that lies below the surface, f(x,y) = 2xy

and above the rectangle $\{(x,y), \mid 0 \le x \le 3,\ 0 \le y \le 2\}$

a. Use a Riemann sum with m = 3 and n = 2 and take the sample point to be the upper
b.
right corner of each rectangle.

b. Use the midpoint rule to estimate the volume of the solid as in part a.

Strategy: First show the rectangular grid and sample points for part a.



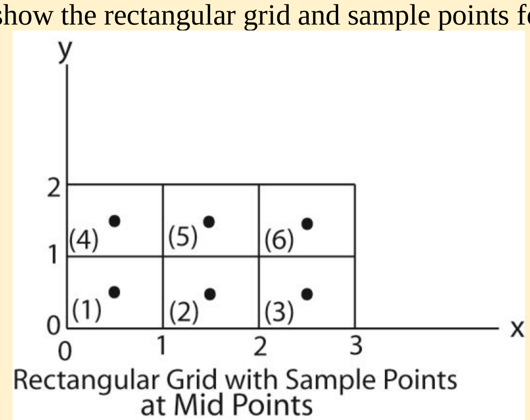
Next set up table to calculate the double Riemann Sum for part a.

	Area	Sample Point Coordinates	f(x*,y*)	Individual Volumes f(x,y)][Area]
1	1	(1,1)	2	2
2	1	(2,1)	4	4
3	1	(3,1)	6	6
4	1	(1,2)	4	4
5	1	(2,2)	8	8
6	1	(3,2)	12	12

$\sum$ of individual volumes = total volume = 36 result

Example: Next use the midpoint rule to estimate the volume of the solid as in part a.
Estimate the volume of a solid S that lies below the surface, f(x,y) = 2xy
and above the rectangle $\{(x,y), | 0 \le x \le 3, 0 \le y \le 2$

Strategy: First show the rectangular grid and sample points for part b.



Next set up table to calculate the double Riemann Sum for part b.

	Area	Sample Point Coordinates	f(x,y) 2xy	Individual Volumes f(x,y) [Area]
1	1	(0.5,0.5)	0.5	0.5
2	1	(1.5,0.5)	1.5	1.5
3	1	(2.5,0.5)	2.5	2.5
4	1	(0.5,1.5)	1.5	1.5
5	1	(1.5,1.5)	4.5	4.5
6	1	(2.5,1.5)	7.5	7.5

$\sum$ of individual volumes = total volume = 18 result