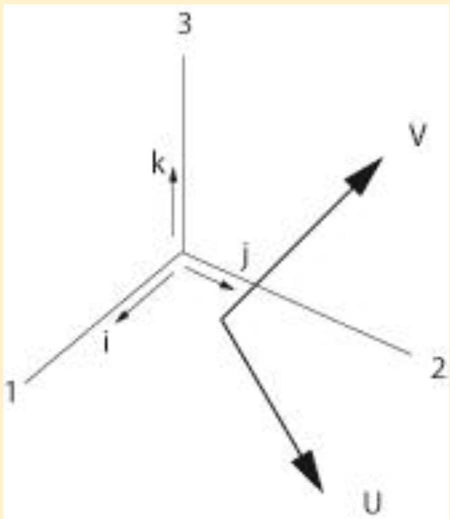
Vectors – Addition, Dot Product, Direction Cosines, Projections

In a Nut Shell: Vectors have magnitude and direction such as velocity and acceleration. Vectors can be added, subtracted, and multiplied. There are two types of vector multiplication. They are the scalar (or dot) product and the vector product.

Vector Addition $\mathbf{U} = (u_1,u_2,u_3)$, $\mathbf{V} = (v_1, v_2, v_3)$



1-2-3 rectangular Cartesian coordinates

Let

u_1 be the component of U along the 1 axis; v_1 be the component of V along the 1 axis

u_2 be the component of U along the 2 axis; v_2 be the component of V along the 2 axis

u_3 be the component of U along the 3 axis; v_3 be the component of V along the 3 axis

Then by vector addition (you add the components):

$$\mathbf{U} + \mathbf{V} = (u_1 + v_1,\; u_2 + v_2,\; u_3 + v_3)$$

Magnitude of a vector $\mathbf{U} = (u_1,u_2,u_3)$

$$U = \sqrt{(u_1^2 + u_2^2 + u_3^2)}$$ (square root of the sum of its squares)

Unit Vector, $\mathbf{e}_U$, is the vector divided by its magnitude. $\mathbf{e}_U = \mathbf{U} / |\mathbf{U}| = \mathbf{U} / U$

Definition of the Base Unit Vectors - i, j, k (along axes 1, 2, 3)

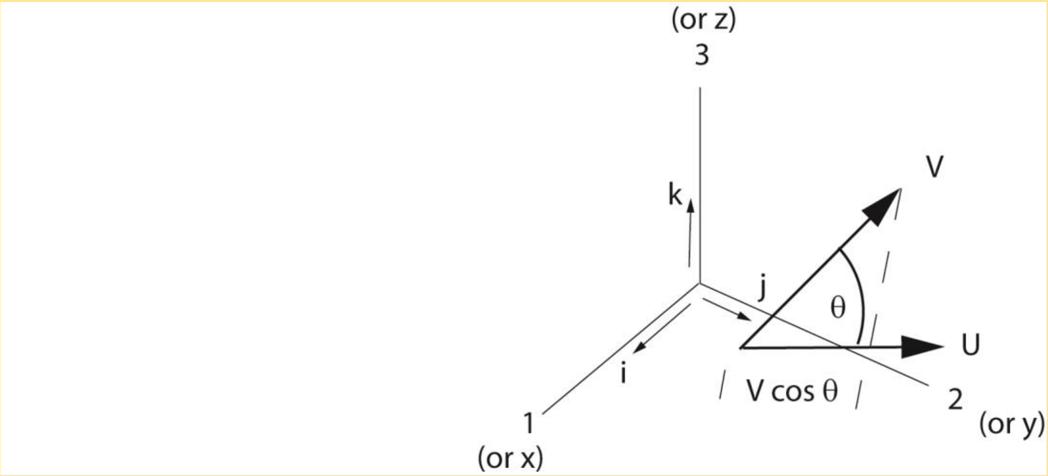
$\mathbf{i} = (1, 0, 0)$ $\mathbf{j} = (0, 1, 0)$, $\mathbf{k} = (0, 0, 1)$

Definition of Dot Product of Two Vectors $\mathbf{U} = (u_1,u_2,u_3)$ and $\mathbf{V} = (v_1, v_2, v_3)$

$\mathbf{U} \cdot \mathbf{V} = (u_1 v_1 + u_2 v_2 + u_3 v_3)$ **The result of the dot product is a scalar value.**

Interpretation of Dot Product Let θ be the angle between **U** and **V** .

$\mathbf{U} \cdot \mathbf{V} = |\mathbf{U}|\ |\mathbf{V}| \cos\theta$ So $\cos\theta = \mathbf{U} \cdot \mathbf{V} / |\mathbf{U}|\ |\mathbf{V}$



One can use this dot product to calculate the angle between a vector **U** and each coordinate axis, x, y, and z. Call them θ_x, θ_y, θ_z . Then the cosine of these angles are called the "direction cosines" . i.e.

$\cos\theta_x = \mathbf{U} \cdot \mathbf{i} / |\mathbf{U}|\ |\mathbf{i}| = u_1 / |\mathbf{U}|$, $\cos\theta_y = \mathbf{U} \cdot \mathbf{j} / |\mathbf{U}|\ |\mathbf{j}| = u_2 / |\mathbf{U}|$

$\cos\theta_z = \mathbf{U} \cdot \mathbf{k} / |\mathbf{U}|\ |\mathbf{k}| = u_3 / |\mathbf{U}|$

Example of a Dot Product

Note: $\mathbf{i} \cdot \mathbf{i} = 1$, $\mathbf{j} \cdot \mathbf{j} = 1$, $\mathbf{k} \cdot \mathbf{k} = 1$, $\mathbf{i} \cdot \mathbf{j} = 0$, $\mathbf{i} \cdot \mathbf{k} = 0$, $\mathbf{j} \cdot \mathbf{k} = 0$

Let $\mathbf{U} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k}$ and $\mathbf{V} = -\,\mathbf{i} + \mathbf{j} - 6\,\mathbf{k}$

$\mathbf{U} \cdot \mathbf{V} = (3)(-1) + (4)(1) + (5)(-6) = -31$ (scalar result)

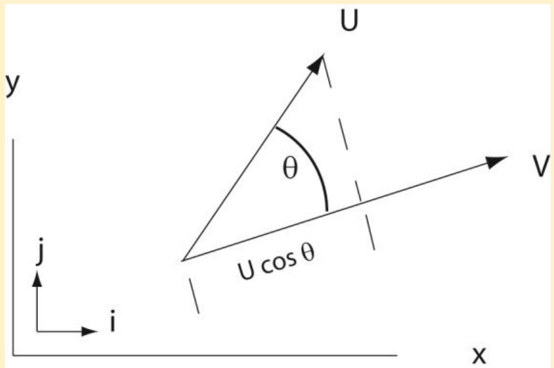
In a Nut Shell: The **scalar projection** of a vector, **U**, in the direction of another vector, **V**, is just the component of **U** along **V**. Its symbol is, $\text{comp}_{\mathbf{V}}\,\mathbf{U}$.

$$\text{comp}_{\mathbf{v}}\,\mathbf{U} = \mathbf{U} \cdot \mathbf{V} / |\mathbf{V}|$$

Note: The result is a scalar. $\mathbf{U} \cdot \mathbf{V} / |\mathbf{V}| = (U) \cos\theta\ (V) / |\mathbf{V}| = U \cos\theta$

Here θ is the angle between the vectors **U** and **V** .

Note: For two vectors in the x-y plane the component of U along V is U cos θ as shown below. A similar result holds for vectors in three dimensions, x-y-z.



In a Nut Shell: The **vector projection** of a vector, **U**, in the direction of another vector, **V**, is just the component of **U** in the direction of **V** times the unit vector along **V.** The symbol for the vector projection of **U** along **V** is $\text{proj}_{\mathbf{V}}\, \mathbf{U}$.

$$\text{proj}_{\mathbf{V}}\, \mathbf{U} = [\, (\mathbf{U} \cdot \mathbf{V}) / |\mathbf{V}|\,]\, \mathbf{e}_{\mathbf{V}}$$

Note: The result is a vector.

A unit vector in the direction of **V** is $\mathbf{V} / |\mathbf{V}| = \mathbf{e}_{\mathbf{V}}$

So $\text{proj}_{\mathbf{V}}\, \mathbf{U} = [\, (U \cos\theta)\,]\, [\, \mathbf{V} / |\mathbf{V}|\,] = [\, (\mathbf{U} \cdot \mathbf{V}) / |\mathbf{V}|\,]\, \mathbf{e}_{\mathbf{V}}$

Example: Find the scalar projection of U and of V given by:

$$\mathbf{U} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{V} = 3\,\mathbf{i} + 4\,\mathbf{j}$$

$$V = \sqrt{[\, (3)^2 + (4)^2\,]} = \sqrt{5}$$

$$\text{comp}_{\mathbf{V}}\, \mathbf{U} = \mathbf{U} \cdot \mathbf{V} \,/\, V = [\, (1)(3) + (1)(4)\,] \,/\, 5 = 7/5 \quad \text{(result)}$$

Example: Find the vector projection of **U** onto **V** given by:

$$\mathbf{U} = \mathbf{i} + \mathbf{j} + \mathbf{k} \quad \text{and} \quad \mathbf{V} = 3\,\mathbf{i} + 4\,\mathbf{j}$$

$$V = \sqrt{[\, (3)^2 + (4)^2\,]} = 5$$

Recall $\text{proj}_{\mathbf{V}}\, \mathbf{U} = (\mathbf{U} \cdot \mathbf{V})\, \mathbf{e}_{\mathbf{V}}$

$$\mathbf{U} \cdot \mathbf{V} = (1)(3) + (1)(4) = 7, \qquad \mathbf{U} \cdot \mathbf{V} \,/\, V = 7/5$$

$$\mathbf{e}_{\mathbf{V}} = (3\,\mathbf{i} + 4\,\mathbf{j}) / 5$$

$$\text{proj}_{\mathbf{V}}\, \mathbf{U} = (\mathbf{U} \cdot \mathbf{V})\, \mathbf{e}_{\mathbf{V}} = (7/25)\, [\, 3\mathbf{i} + 4\mathbf{j}\,] \quad \text{(result)}$$

Example: Determine which of the following expressions are meaningful.

1. $(\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$ Not meaningful $\mathbf{A} \cdot \mathbf{B}$ is a scalar and scalar product is between two vectors.

2. $(\mathbf{A} \cdot \mathbf{B})\, \mathbf{C}$ Meaningful since $\mathbf{A} \cdot \mathbf{B}$ is a scalar and you can multiply a scalar times a vector.

3. $(\mathbf{A} \cdot \mathbf{B}) + \mathbf{C}$ Not meaningful since $\mathbf{A} \cdot \mathbf{B}$ is a scalar and $\mathbf{C}$ is a vector