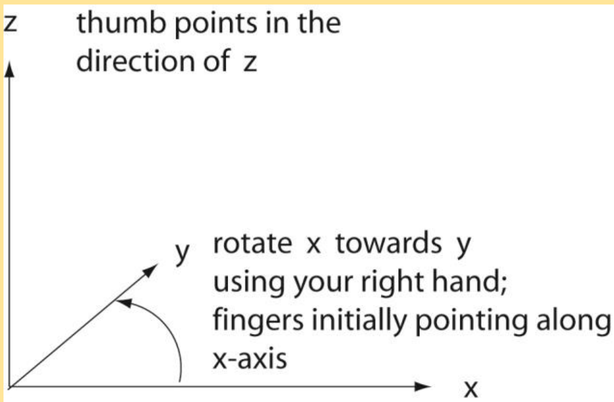
Vector and Scalar Triple Products

In a Nut Shell: The **vector product** of two vectors results in a new vector perpendicular to original two vectors. The direction of the new vector is conveniently determined using the **"right hand rule"** as described below.

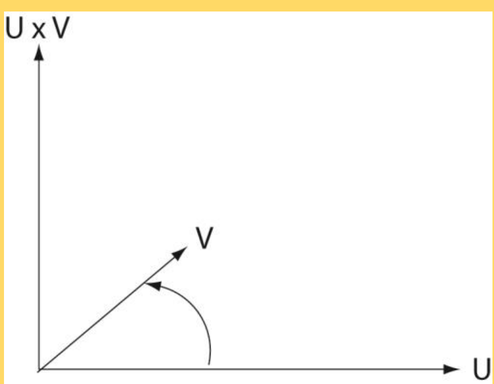
For a rectangular coordinate system point the fingers of your right hand in the direction of the x-axis. Then rotate your fingers towards the y-axis. The result of a vector product is another vector (your thumb) perpendicular to the two original vectors (along the z-axis).



The **vector product, also called the cross product**, of $\mathbf{U} = (u_1, u_2, u_3)$ and of $\mathbf{V} = (v_1, v_2, v_3)$ is define by the 3 x 3 determinant below as:

$$\mathbf{U} \times \mathbf{V} = \det \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{matrix}$$

with **i** , **j** , **k** in the first row u_1, u_2, u_3 in the second row, and v_1, v_2, v_3 in the third row



Example: Evaluate the vector product, **U** x **V**, for the vectors

$$\mathbf{U} = 3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k} \quad \text{and} \quad \mathbf{V} = \mathbf{i} - 2\,\mathbf{j} - 3\,\mathbf{k}$$

Use the "right hand rule" to rotate the vector **U** into **V**. The resultant of this product is a new vector, **W**, perpendicular to **U** and **V**.

Strategy: Use a 3 x 3 determinant (det) to calculate the vector product of **U** and **V** with **i**, **j**, and **k** in the first row, **U** in the second row and **V** in the third row.

$$\mathbf{W} = \mathbf{U} \times \mathbf{V} = \det \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 5 \\ 1 & -2 & -3 \end{vmatrix}$$

$$\mathbf{U} \times \mathbf{V} = \mathbf{i}\det \begin{vmatrix} 4 & 5 \\ -2 & -3 \end{vmatrix} - \mathbf{j}\det \begin{vmatrix} 3 & 5 \\ 1 & -3 \end{vmatrix} + \mathbf{k}\det \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix}$$

$$\mathbf{U} \times \mathbf{V} = \mathbf{i}\,[(4)(-3) - (-2)(5)] - \mathbf{j}\,[(3)(-3) - (1)(5)] + \mathbf{k}\,[(3)(-2) - (1)(4)]$$

$$\mathbf{W} = -2\,\mathbf{i} + 14\,\mathbf{j} - 10\,\mathbf{k}$$ (result for vector product)

Note: **W** is a new vector that is perpendicular to vectors **U** and **V** so the dot product of **W** with both **U** and **V** should be zero.

Check: $\mathbf{W} \cdot \mathbf{U} = (-2\,\mathbf{i} + 14\,\mathbf{j} - 10\,\mathbf{k}) \cdot (3\,\mathbf{i} + 4\,\mathbf{j} + 5\,\mathbf{k})$

$\mathbf{W} \cdot \mathbf{U} = (-2)(3) + (14)(4) + (-10)(5) = 0$

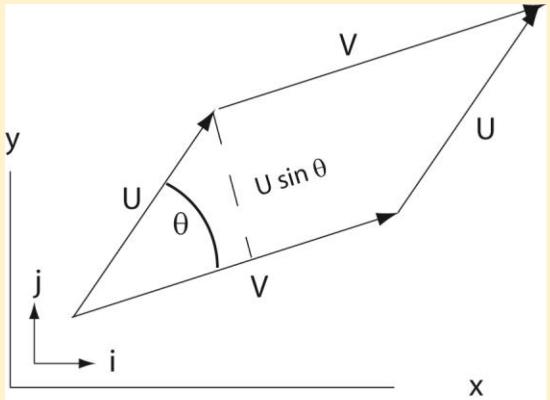
Also $\mathbf{W} \cdot \mathbf{V} = (-2)(1) + (14)(-2) + (-10)(-3) = 0$

Magnitude of the Vector Product (Cross Product) - is a scalar

$|\mathbf{U} \times \mathbf{V}| = |\mathbf{U}||\mathbf{V}| \sin\theta$ where θ is the angle between vectors **U** and **V**

Note: $|\mathbf{U} \times \mathbf{V}|$ represents the area of a parallelogram with sides **U** and **V** where the angle between **U** and **V** is θ

See the figure below.

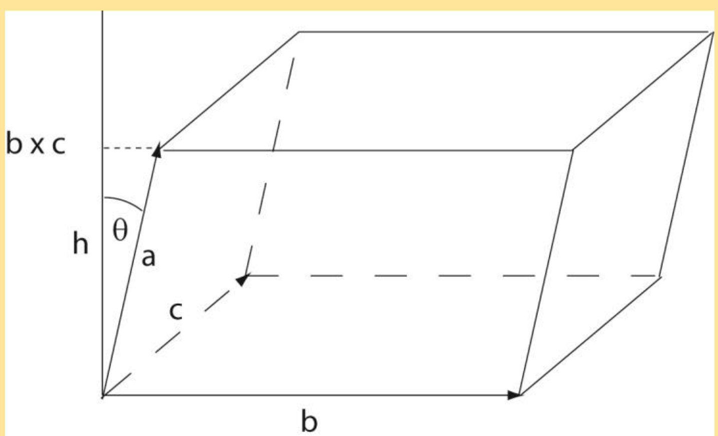


The Scalar Triple Product, T, is defined as follows:

$T = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ **Note:** the result is a scalar $|\mathbf{b} \times \mathbf{c}| = h$

$$T = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

This triple product yields the volume of a parallelepiped with sides **a** , **b, and c**



Also, the volume of a pyramid (sides **a**,**b,** and **c**) is 1/6 the volume of the parallelepiped.