Heat Conduction in a Thin Rod

In a Nut Shell: Heat conduction in a thin rod is governed by the following partial differential equation:

$$\partial u/\partial t \;=\; k\, \partial^2 u/\partial x^2 \quad \text{----------------------} \quad (1)$$

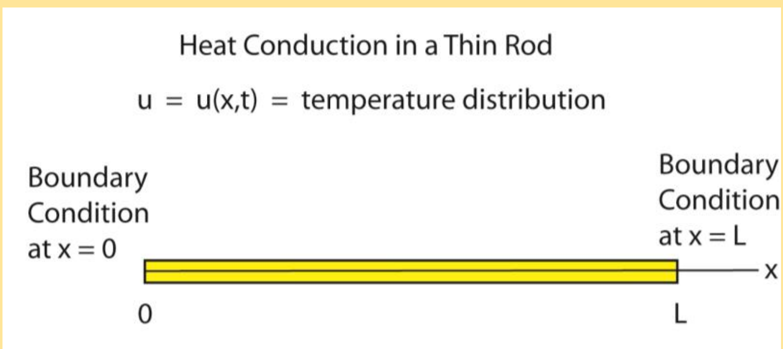
where $u = u(x,t)$ = the temperature distribution in the rod
x = the position along the rod
t = the time at which the temperature at x is $u(x,t)$

and

k is the thermal diffusivity of the rod (material property)

The desired outcome is to predict the temperature distribution, $u(x,t)$, in the rod subject to the boundary (end) conditions given an initial temperature distribution in the rod, $u(x,0) = f(t)$.

Since the partial differential equation is second order in its derivative with respect to x, you will need two boundary conditions. The p.d.e is first order in its derivative with respect to t, so you need one initial condition. See the figure below.



The common boundary conditions at the ends of the rod are as follows:

a. Specified temperature at an end $x = 0$ — $u(0,t) = To$
b. Specified temperature at end $x = L$ — $u(L,t) = T_1$
c. Insulated condition at $x = 0$ — $\partial u(0,t)/\partial x = 0$
d. Insulated condition at $x = L$ — $\partial u(L,t)/\partial x = 0$

or any combination of these boundary conditions. i.e. If the temperature is specified at $x = L$ and the rod is insulated at $x = 0$, then the appropriate boundary conditions are $\partial u(0,t)/\partial x = 0$ and $u(L,t) = 0$. Other linear combinations might be like.

$$C_1\, u(0,t) + C_2\, u_x(0,t) = F(t) \quad \text{and} \quad C_3\, u(L,t) + C_4\, u_x(L,t) = G(t)$$

where C_1, C_2, C_3, and C_4 are known constants and where $u_x = \partial u/\partial x$.

Strategy: Since the heat conduction equation involves two independent variables, x and t, apply the method of "separation of variables" to separate out the spatial variable, x, from the time variable, t.

Method of Separation of Variables:

Assume $u(x,t) = X(x)\,T(t)$ (for separation of variables)

Put this expression into the heat conduction equation,

$$\partial u/\partial t = k\, \partial^2 u/\partial x^2$$

Take the value for the thermal conductivity of the rod, $k = 1$.
The table below contains the calculations.

$$X\, dT/dt = d^2X/dx^2\, T$$

Let $d^2X/dx^2 = X''$ and $dT/dt = T^*$

Then divide both sides by $X\,T$ to separate the variables:

$$T^*/T = X''/X$$

Since T^*/T depends only on t and X''/X depends only on x, the only way

T^*/T could equal X''/X is for both to be constant.

Let K = constant = separation constant = $-\lambda$

So $T^*/T = X''/X = -\lambda$

Result: For solution of heat conduction in a thin rod, you need to solve an

eigenvalue problem of the form:

$X'' + \lambda X = 0$
$T^* + \lambda T = 0$

Example: Find the temperature distribution in a thin rod for:

$$\partial u/\partial t = 2\, \partial^2 u/\partial x^2 \quad \text{---} \quad (1)$$

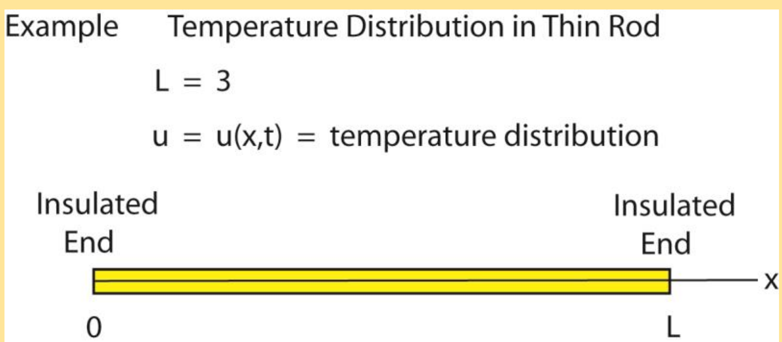
$u_x(0,t) = u_x(3,t) = 0$ (boundary conditions)

$u(x,0) = 4\cos(2\pi x/3) - 2\cos(4\pi x/3)$ (initial temperature distribution)

Note: The boundary conditions listed above simulate insulation (no heat transfer) at each end of the thin rod. See the figure below.

The initial condition represents the temperature distribution throughout the thin rod at the beginning, t = 0.

The objective is to find the distribution of temperature, u(x,t), in the thin rod for any location, x, and for any time, t.



Strategy: Apply separation of variables.

Let $u(x,t) = X(x)\,T(t)$, $u_t = XT'$, $u_{xx} = X''$ put into (1) above

So $XT' = 2X''T$ or $X''/X = T'/2T = K$ = separation constant $= -\lambda$

$$X'' + \lambda X = 0 \quad \text{and} \quad T' + 2\lambda T = 0$$

This eigenvalue problem has three possible eigenvalues. λ could be positive, negative, or zero.

Case 1: $\lambda = 0$ Then $X'' = 0$ and $X(x) = Ax + B$, $X'(x) = A$

$X'(0) = 0$ gives $A = 0$ leaving $X(x) = B$ $X'(x) = 0$

Result for case 1: $\lambda = 0$ (eigenvalue) and $X_o(x) = 1$ (eigenvector)

Case 2: $\lambda < 0$ Let **:** $\lambda = -\alpha^2$ then

$X'' - \alpha^2 X = 0$ so $X(x) = A \cosh \alpha x + B \sinh \alpha x$ and

$X'(x) = \alpha A \sinh \alpha x + \alpha B \cosh \alpha x$

$X'(0) = \alpha B$ and since $\alpha \neq 0$ $B = 0$

$X'(3) = \alpha A \sinh 3\alpha$ but $\alpha \neq 0$ and $\sinh 3\alpha \neq 0$ $A = 0$

So $A = B = 0$ so $X(x) = 0$

Result for Case 2: No eigenvalues nor eigenvectors.

Case 3: $\lambda > 0$ Let **:** $\lambda = \alpha^2$ then (λ are eigenvalues)

$X'' + \alpha^2 X = 0$ where $X'(0) = X'(3) = 0$

$X = A \cos \alpha x + B \sin \alpha x$

$X' = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$

$X'(0) = B\alpha = 0$ so $B = 0$ since $\alpha \neq 0$

$X'(3) = -A\alpha \sin 3\alpha = 0$ For a nontrivial solution $A \neq 0$

So $3\alpha = n\pi$ or $\alpha = n\pi/3$

Result for Case 3: $\lambda = (n\pi/3)^2$ (eigenvalues)

and

$X_n(x) = \cos(n\pi x/3)$ (eigenfunctions)

Next solve the d.e. involving time.

$T' + 2\alpha^2 T = 0$ where $\alpha^2 = (n\pi/3)^2$

Assume $T(t) = C e^{rt}$ and $T' = C r e^{rt}$

Substitute these expressions into the d.e.

Then $C e^{rt} [r + 2\alpha^2] = 0$, $r = -2\alpha^2$

$T_n(t) = \exp[-2(n\pi/3)^2 t] = \exp[-2n^2\pi^2 t/9]$

Recall that the "product" solution has the form

$$u_n(x,t) = T_n(t)\, X_n(x)$$

which gives $u_n(x,t) = \exp[-\,2n^2\pi^2\, t / 9\,] \cos(n\pi x/3)$

and recall $u_o(x,t) = 1$

So the trial solution for the temperature distribution in the rod is:

$$u(x,t) = a_o + \sum_{n=1}^{\infty} a_n \exp[-\,2n^2\pi^2\, t/9] \cos(n\pi x/3) \qquad \text{----------------------------} \quad (2)$$

Next apply the initial condition (the initial temperature distribution) that

$$u(x,0) = 4\cos(2\pi x/3) - 2\cos(4\pi x/3)$$

$$u(x,0) = a_o + \sum_{n=1}^{\infty} a_n \cos(n\pi x/3) = 4\cos(2\pi x/3) - 2\cos(4\pi x/3)$$

Since the initial temperature distribution involves $\cos(2\pi x/3)$ and $\cos(4\pi x/3)$

the only terms in the Fourier cosine series that are nonzero should be for

$n = 2$ and $n = 4$ with $a_2 = 4$ and $a_4 = -\,2$ To show this calculate the

Fourier coefficients. This calculation is shown below.

P = period = 6, P = 2L where L = 3

$$a_o = (1/3)\int_0^6 4\cos(2\pi x/3) - 2\cos(4\pi x/3)\, dx = 0$$

$$a_n = (1/3)\int_0^6 [4\cos(2\pi x/3) - 2\cos(4\pi x/3)] \cos(n\pi x/3)\, dx$$

Next let $u = \pi x/3$, $du = \pi\, dx/3$ and apply the orthogonality conditions:

$$a_n = (1/\pi)\int_{u=0}^{u=2\pi} [4\cos(2u) - 2\cos(4u)] \cos nu\, du = \begin{cases} 0 & \text{if } n \neq 2 \text{ or } 4 \\ \pi & \text{if } n = 2 \text{ or } 4 \end{cases}$$

$$a_2 = (1/\pi)\int_{u=0}^{u=2\pi} [4\cos(2u)] \cos nu\, du = 4 \qquad \text{if } n = 2$$

$$a_4 = (1/\pi)\int_{u=0}^{u=2\pi} [-2\cos(4u)] \cos nu\, du = -\,2 \qquad \text{if } n = 4$$

Put these coefficients into (2) for the final temperature distribution.

$$u(x,t) = 4\exp[-8\pi^2 t/9]\cos(2\pi x/3) - 2\exp[-32\pi^2 t/9]\cos(4\pi x/3) \qquad \text{(result)}$$

Example: A thin copper 50 cm long with both ends and lateral surfaces insulated. The initial temperature in the rod $u(x,0) = 2x$.

Find the temperature in the rod $u(x,t)$.

Determine the temperature at $x = 10$ cm after 1 min, and also the time it will take for the temperature at $x = 10$ cm to reach 45° C.

For one dimensional heat conduction in a thin rod:

$$\partial u/\partial t = k\, \partial^2 u/\partial t^2 \quad \text{--------------------------------} \ (1)$$

here $u = u(x,t)$ = the temperature in the rod
x = the position in the rod
t = the time at which the temperature at x is $u(x,t)$

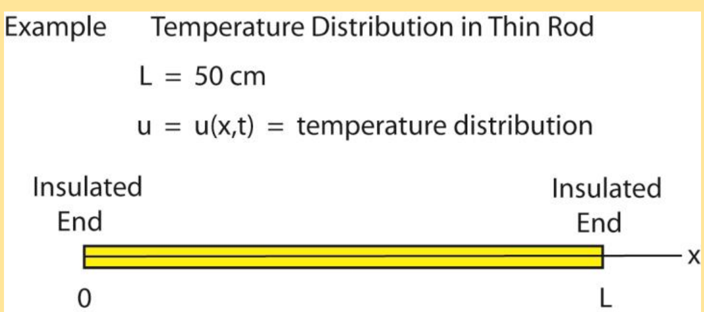
and
k is the thermal diffusivity of copper = 1.15 cm^2/sec (from Table)

Let each end be insulated (no heat transfer). Then,

$$u_x(0, t) = u_x(50, t) = 0 \quad \text{------------------------------} \ (2)$$

Let the initial temperature distribution be linear such as:

$$u(x, 0) = 2x \quad \text{---} \ (3)$$



Use separation of variables approach to find eigenvalues and eigenfunctions.

So assume $u(x,t) = X(x)\, T(t)$ and substitute into eq.(1).

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Strategy: Start with $d^2X/dx^2 = X''$ and $dT/dt = T'$ So separation of variables gives

$$X\, dT/dt = k\, d^2X/dx^2\, T \quad ; \quad X''/X = T'/kT = \text{constant} = -\lambda$$

Recall for $\lambda = 0$ (from the first example) $\lambda_o = 0$ and $X_o(x) = 1$

and there were no eigenvalues nor eigenfunctions if $\lambda < 0$.

So the case of interest is for $\lambda > 0$, $\lambda = \alpha^2$

So $X'' + \alpha^2 X = 0$ subject to $X'(0) = X'(50) = 0$

$$X = A \cos \alpha x + B \sin \alpha x; \quad X' = -\alpha A \sin \alpha x + \alpha B \cos x$$

$X'(0) = 0 = \alpha B$; since $\alpha \neq 0$, $B = 0$

$X'(50) = 0 = -\alpha A \sin 50\alpha$; since $\alpha \neq 0$, for a nontrivial solution $B \neq 0$

Hence $\sin 50\alpha = 0$ and $50\alpha = n\pi$ $n = 1, 2, 3,$ $\alpha = n\pi/50$

and $\lambda = n^2\pi^2/2500$ = eigenvalues;

$X_n = \cos(n\pi x/50)$ = eigenfunctions

Strategy: Use these eigenvalues for the time dependent relation:

$T' + k\alpha^2 T = 0$ Assume an exponential solution: $T = C e^{rt}$

So $C e^{rt} (r + k\alpha^2) = 0$ and $r = -k\alpha^2$

$$T_n(t) = \exp(-n^2\pi^2 kt/2500 = \exp[-n^2\pi^2 (1.15)t/2500]$$

Next apply the "product" solution $u_n(x,t) = T_n(t) X_n(x)$

$$u_n(x,t) = \exp[-n^2\pi^2 (1.15)t/2500] \cos(n\pi x/50)$$

$$u(x,t) = a_o + \sum_{n=1}^{\infty} an \exp[-n^2\pi^2 (1.15)t/2500] \cos(n\pi x/50)$$

The initial condition gives $u(x, 0) = 2x = a_o + \sum_{n=1}^{\infty} an \cos(n\pi x/50)$

Strategy: Calculate the Fourier cosine coefficients.

$$a_o = (2/50)\int_0^{50} 2x\,dx = 100$$

$$a_n = (2/50)\int_0^{50} 2x\cos(n\pi x/50)dx = 100 \quad \text{(integrate by parts)}$$

$$u = (4/50)x \qquad dv = \cos(n\pi x/50)$$

$$du = (4/50)dx \qquad v = (50/n\pi)\sin(n\pi x/50)$$

$$a_n = (4/n\pi)\sin(n\pi x/50)\Big|_0^{50} - \int_0^{50}(4/n\pi)\sin(n\pi x/50) = -\int_0^{50}(4/n\pi)\sin(n\pi x/50)$$

$$a_n = (4/n\pi)(50/n\pi)\cos(n\pi x/50)\Big|_0^{50} = 200/(/n^2\pi^2)\,[\cos n\pi - 1]$$

$$a_n = 200/(/n^2\pi^2)\,[(-1)^n - 1]$$

So $a_n = 0$ for n even and $-400/(/n^2\pi^2)$ for n odd

$$u(x,t) = 100/2 - \sum_{n\ odd} 400/(/n^2\pi^2)\ \exp[-n^2\pi^2(1.15)t/2500]\cos(n\pi x/50) \quad \text{(result)}$$

Next calculate the temperature in the rod at x = 10 cm after one minute.

$$u(x,t) = 100/2 - \sum_{n\ odd} 400/(/n^2\pi^2)\exp[-n^2\pi^2(1.15)t/2500]\cos(n\pi x/50)$$

Now let $x = 10$, $t = 1$ min $= 60$ sec

Take a one-term approximation since exponential term drops rapidly.

$$u(10, 60) = 50 - 400/(/\pi^2)\exp[-\pi^2(1.15)60/2500]\cos(n\pi/5)$$

$u(10, 60) = 50 - 24.9698 = 25.03^o$ C If we take a second term (n = 3)

$$\text{Term } 2 = 400/(/9\pi^2)\exp[-9\pi^2(1.15)60/2500]\cos(3\pi/5)$$

$$\text{Term } 2 = +0.1199$$

So $u(10, 60) = 50 - 24.9698 + 0.1199 = 25.15^o$ C

Finally, determine the time when the temperature is 45^o C at $x = 10$ in the rod.

where

$$u(x,t) = 100/2 - \sum_{n \text{ odd}} 400/(/n^2\pi^2) \exp - n^2\pi^2 (1.15)\, t /2500]\cos(n\pi x/50)$$

Approximation: Consider one term only. (n = 1)

$$u(10, t^*) = 45 = 50 - 400/(/\pi^2) \exp[-\pi^2 (1.15)\, t^* /2500] \cos(\pi/5)$$

$$- 5 = - 400/(/\pi^2) \exp[-\pi^2 (1.15)t^* /2500] \cos(\pi/5)$$

$$[5\pi^2 /400] / \cos(\pi/5) = \exp[- \pi^2 (1.15)\, t^* /2500]$$

Now in order to solve for t^* , take the logarithm of both sides.

$$\ln [5\pi^2 /400 \cos(\pi/5)] = - \pi^2 (1.15)t^* /2500$$

$$t^* = (2500/1.15\pi^2) \ln\{400\cos(\pi/5)/ 5\pi^2\} = (2500/1.15\pi^2) \ln \{80\cos(\pi/5)/\pi^2 \}$$

t^* = 414.23 sec = 6 min 54 sec (result)