Absolute Maxima and Minima of Functions with Two Variables

In a Nut Shell: Absolute maximum and minimum values of functions in the case of two independent variables, (x,y), occur at the "critical locations" within or on the boundary of the domain, D.

A **local maximum** occurs near (a,b) if $f(x,y) \le f(a,b)$. Likewise a **local minimum** occurs near (a,b) if $f(x,y) \ge f(a,b)$. An **absolute maximum** occurs at (a,b) if $f(x,y) \le f(a,b)$ for all points (x,y) in the domain of the function and an **absolute minimum** occurs at (a,b) if $f(x,y) \ge f(a,b)$ for all points (x,y) in the domain of the function.

You must investigate both interior points and points on the boundary of the domain.

Locating critical points of a function, f(x,y on the interior of the domain):

The slope of the function, f(x,y), must be zero at each critical point, (a,b). For functions of two independent variables, (x,y), the following partial derivatives must hold:

$$\partial f(a,b)/\partial x = 0 \quad \text{and} \quad \partial f(a,b)/\partial y = 0$$

Strategy for finding Absolute Minimum, Absolute Maximum of f(x,y)

Step 1	Locate critical points, (a,b) within the domain	$\partial f(a,b)/\partial x = 0$ and $\partial f(a,b)/\partial y = 0$
Step 2	Calculate f(a,b) for each critical point within in the domain, D.	Locates local maxima and local Minima.
Step 3	Calculate f(x,y) on the boundaries of the domain	Find extreme values of f(x,y) on the boundaries of the domain, D, by setting the first derivative of the function along the boundary to zero and calculating the extreme values at the critical points on the boundary.

The largest value from step 2 and 3 is the **absolute maximum** and the smallest value from step 2 and step 3 is the **absolute minimum**.

Example: Find the absolute maxima and minima for the function, f(x,y) given below For the domain $|x| \le 1$ and $|y| \le 1$.

$$f(x,y) = x^2 + y^2 + x^2y + 9$$

Strategy:

Locate critical points (a,b)

and

$f/\partial x = 2x + 2xy = 0 = 2x(1 + y)$ (equation 1)

$\partial f/\partial y = 2y + x^2 = 0$ (equation 2)

Use equation 1. By equation 2 $y = 0$

Case 1: $x = 0$ Critical point is at (0,0)

Use equation 1. By equation 2 $x = \pm\sqrt{2}$

Case 2: $y = -1$ Critical points are at: $(\sqrt{2}, -1)$ and $(-\sqrt{2}, -1)$ (not in domain)

Calculate D(a,b) $f_{xx} = 2(1 + y)$, $f_{xy} = 2$, $f_{xy} = 2x$ $D = f_{xx}f_{yy} - f_{xy}^2$

Strategy:

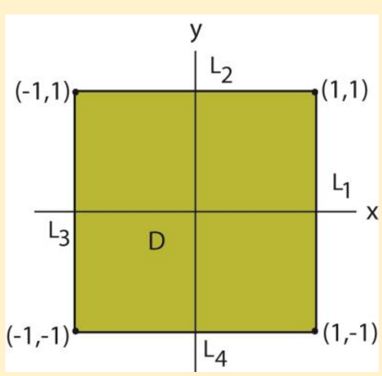
Set up table to identify local max, local min, and saddle points for f(x,y)

Results

x,y	f_{xx}	f_{yy}	f_{xy}	D	Type
0,0	2	0	2	4 > 0	minimum

Next evaluate values of f(x,y) on the boundaries of domain, D.

Strategy: Next calculate the maxima and minima of the function on the boundaries of the domain. Then compare the values found on the boundary with the values within the domain to find the global (absolute) maxima or minima of the function. The figure below shows the domain of the function and its boundaries.



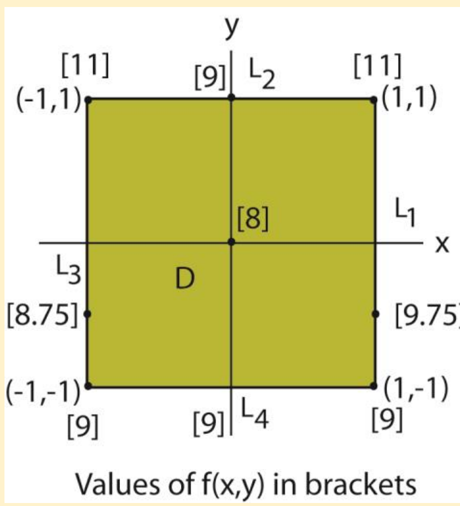
$$f(x,y) = x^2 + y^2 + x^2 y + 8$$

Critical Points on Boundaries of Domain

					Critical Point
Along L_1	$x = 1$	$-1 \le y \le 1$	$f(y) = y^2 + y + 9$	$df/dy = 2y + 1$	$y = -1/2$
Along L_2	$y = 1$	$-1 \le x \le 1$	$f(x) = 2x^2 + 9$	$df/dx = 4x$	$x = 0$
Along L_3	$x = -1$	$-1 \le y \le 1$	$f(y) = y^2 + y + 9$	$df/dy = 2y + 1$	$y = -1/2$
Along L_4	$y = -1$	$-1 \le x \le 1$	$f(x) = 9$	$df/dx = 0$	Constant on L_4

$f(1, -1/2) = 9.75$, $f(0,1) = 9$, $f(-1, -1/2) = 9.75$, $f(\text{any } x, -1) = \text{constant} = 9$

Location and values for Maxima and Minima



Absolute Maxima: $f_{max} = 11$ at $(-1,1)$ and $(1,1)$

Absolute Minima: $f_{max} = 8$ at $(0,0)$