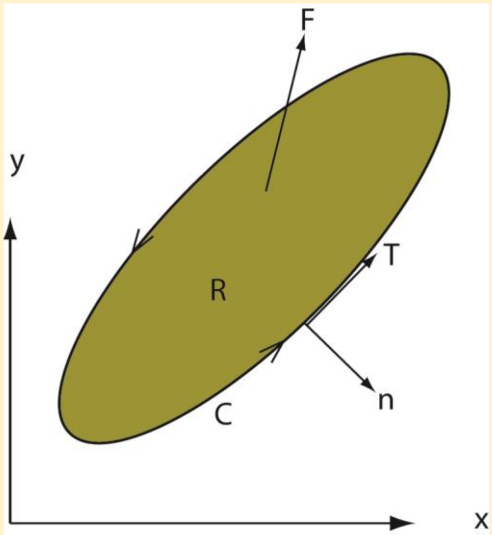
Divergence Theorem

Background: Recall that Green's theorem gives the relationship between a line integral around a simple closed curve, C, and a double integral over the plane region R bounded by *C*. See the figure below.



Also recall that **the "divergence form" of Green's theorem** is

$$\int_C \mathbf{F} \cdot \mathbf{n} \; ds = \iint_R \operatorname{div} \mathbf{F} \; dA$$

where R is a region in the x-y plane enclosed by a piecewise-smooth, positively oriented (keep region to your left as you travel around the simple closed curve C)

F(x,y) is a vector field $\quad \mathbf{F} = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$,
n(x,y) is a unit vector to the curve C
ds = arc length along curve C
div **F** = $\partial P/\partial x + \partial Q/\partial y$
dA = element of area in R

In a Nut Shell: The Divergence Theorem extends the divergence form of Green's theorem from two to three dimensions.

In this case the line integral around a closed curve, C, is replaced by a surface integral around a closed surface, S, and the area integral involving the divergence of the vector field **F** is replaced by the volume integral of the divergence of the vector field, **F** .

So we go from $\int_C \mathbf{F} \cdot \mathbf{n} \; ds = \iint_R \text{div}\, \mathbf{F} \, dA$	**Green's Theorem**
to $\iint_S \mathbf{F}(x,y,z) \cdot \mathbf{n} \; dS = \iiint_E \text{div}\, \mathbf{F}(x,y,z) \; dV$	**Divergence Theorem**

Here $\mathbf{F} = P(x,y,z)\, \mathbf{i} + Q(x,y,z)\, \mathbf{j} + R(x,y,z)\, \mathbf{k}$

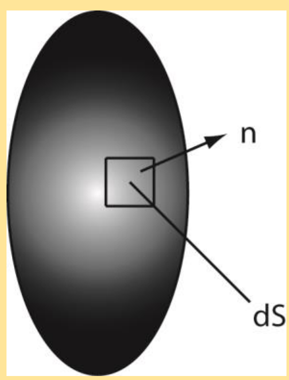
$\mathbf{n}$ = unit normal to the closed surface S

dS = element of area on surface S

$\text{div}\, \mathbf{F} = \partial P/\partial x + \partial Q/\partial y + \partial R/\partial z$

dV = element of volume for the solid region E

E = volume of solid region



Alternate Strategies in the application of the Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{dS} = \iint_S \mathbf{F} \cdot \mathbf{n} \; dS = \iiint_E \text{div}\, \mathbf{F} \, dV$$

F(x,y,z) = vector field, **dS** = element of oriented surface, **n** = unit normal to S

dV = element of volume of region E

Method 1: Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS$ directly over the surface S

Method 2:

Transform the double integral on S $\iint_S \mathbf{F} \bullet \mathbf{n} \, dS$

to a surface integral using $\mathbf{n} = (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})|$

which gives

$$\iint_S \mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) / |(\mathbf{r_u} \times \mathbf{r_v})| dS$$

and $dS = |(\mathbf{r_u} \times \mathbf{r_v})| \, dA$

to obtain the final result

$$\iint_S \mathbf{F} \bullet \mathbf{n} \, dS = \iint_R \mathbf{F} \bullet (\mathbf{r_u} \times \mathbf{r_v}) \, dA$$

where dA is the element of area on R, on the u-v plane

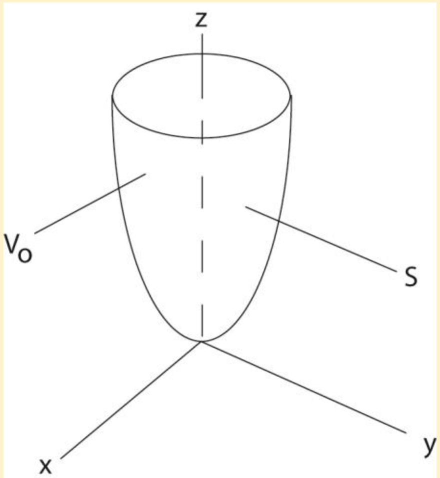
Method 3:

Evaluate $\iiint_E \text{div} \, \mathbf{F} \, dV$ directly over the volume E

Side note: Some mathematicians refer to the Divergence Theorem as Gauss' Theorem.

Example: The force, **F**, acts on the boundary, S, of a closed surface bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$. Use the Divergence Theorem to evaluate the "flux" of **F** acting over the boundary of S **.**

Note: flux = $\iint_S \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{Vo} \text{div} \, \mathbf{F} \, dV$ See the figure below.



$$\text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{Vo} \text{div}\,\mathbf{F}\, dV$$

With $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + 3\,\mathbf{k}$, $\text{div}\,\mathbf{F} = 1 + 1 + 0 = 2$

So $\text{flux} = \iiint_{Vo} 2\, dV$ Next use cylindrical coordinates to evaluate this volume integral.

$$dV = r\, dr\, d\theta\, dz$$

$$\text{flux} = 2 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r^2}^{z=4} dz\, r\, dr\, d\theta = 2 \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} (4 - r^2)\, r\, dr\, d\theta =$$

$$\text{flux} = 2 \int_{\theta=0}^{\theta=2\pi} (4r^2/2 - r^4/4)\Big|_0^2\, d\theta = 2 \int_{\theta=0}^{\theta=2\pi} 4\, d\theta = 16\pi \qquad \text{(result for flux)}$$

Example: Show that the "flux" of the vector field $\mathbf{F} = (3y \cos z)\mathbf{i} + (x^2 e^z)\,\mathbf{j} + (x \sin y)\,\mathbf{k}$

is zero over any closed surface, S, enclosing a solid region, Vo.

$$\text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{Vo} \text{div}\,\mathbf{F}\, dV$$

Now $\text{div}\,\mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z = (0) + (0) + (0) = 0$

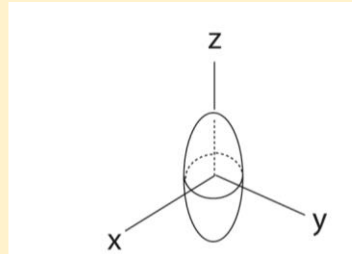
So $\text{flux} = \iiint_{Vo} (0)\, dV = 0$ (result)

Example: Use the Divergence Theorem to evaluate the surface integral

$\int\int_S \mathbf{F} \cdot \mathbf{n} \, dS$ where $\mathbf{F} = xy\,\mathbf{i} + xz\,\mathbf{j} + yz\,\mathbf{k}$

and $\mathbf{n}$ is the outward normal to the ellipsoid

$x^2 + 4y^2 + z^2 = 1$ (Figure on the right)



Thus this surface integral, $I = \int\int_S \mathbf{F} \cdot \mathbf{n} \, dS = \int\int\int_{Vo} \text{div}\,\mathbf{F} \, dV$

$\text{div}\,\mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z = y + y = 2y$

$$\text{So } I = \int\int\int_{Vo} 2y \, dV = \int_{x=-1}^{x=1} \int_{y=-(1/2)\sqrt{(1-x^2)}}^{y=(1/2)\sqrt{(1-x^2)}} \int_{z=-(1/2)\sqrt{(1-x^2-4y^2)}}^{z=(1/2)\sqrt{(1-x^2-4y^2)}} 2y \, dz \, dy \, dx$$

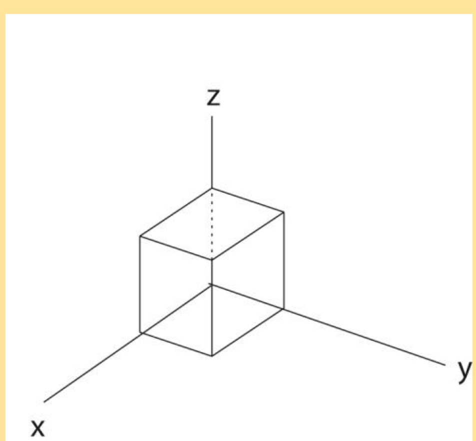
Now notice that if $\mathbf{F} = -xy\,\mathbf{i} + xz\,\mathbf{j} - yz$

Then $\text{div}\,\mathbf{F} = -2y = -\text{div}\,\mathbf{F}$; since the ellipsoid is symmetrical about the plane $y = 0$, the value of the surface integral is zero. Direct evaluation of the surface integral yields the same result.

Example: Find the value of the flux of the vector field

$\mathbf{F} = 2xy\,\mathbf{i} + z^2y\,\mathbf{j} + xz\,\mathbf{k}$ over the unit cube formed by the coordinate planes

and the planes $x = 1$, $y = 1$, $z = 1$ See the figure below.



$$\text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_{Vo} \text{div}\, \mathbf{F}\, dV$$

Now $\text{div}\, \mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z = 2y + z^2 + x$

So $$\text{flux} = \int_{x=0}^{x=1} \int_{y=0}^{y=1} \int_{z=0}^{z=1} (2y + z^2 + x)\, dz\, dy\, dx$$

$$\text{flux} = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (2yz + z^3/3 + xz)\Big|_0^1\, dz\, dy\, dx = \int_{x=0}^{x=1} \int_{y=0}^{y=1} (2y + 1/3 + x)\, dy\, dx$$

$$\text{flux} = \int_{x=0}^{x=1} (y^2 + y/3 + xy)\Big|_{y=0}^{y=1}\, dx = \int_{x=0}^{x=1} (1 + 1/3 + x)\, dx = 1 + 1/3 + 1/2$$

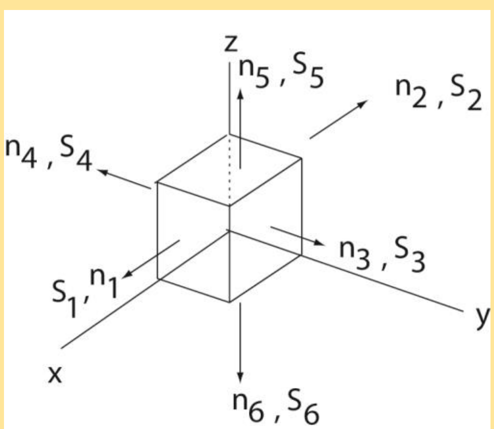
$$\text{flux} = 6/6 + 2/6 + 3/6 = 11/6 \qquad \text{(result)}$$

Alternate solution as a check:

Example: Find the value of the flux of the vector field $\mathbf{F} = 2xy\, \mathbf{i} + z^2y\, \mathbf{j} + xz\, \mathbf{k}$

over the unit cube formed by the coordinate planes and the planes $x = 1$, $y = 1$, $z = 1$

by calculating the surface integral, $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$. See the figure below.



$$\text{flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \int\!\!\int_{S_1 + \; + \; S_6} \mathbf{F} \cdot \mathbf{n}\, dS \qquad \text{where}$$

$\mathbf{n}_1 = \mathbf{i}$, $\mathbf{n}_2 = -\mathbf{i}$, $\mathbf{n}_3 = \mathbf{j}$, $\mathbf{n}_4 = -\mathbf{j}$, $\mathbf{n}_5 = \mathbf{k}$, $\mathbf{n}_6 = -\mathbf{k}$

$dS_1 = dS_2 = dydz$, $dS_3 = dS_4 = dxdz$, $dS_5 = dS_6 = dxdy$

$$\text{flux} = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} 2xy \, dydz = \int_{y=0}^{y=1} \int_{z=0}^{z=1} 2y \, dydz = 1$$

Since $x = 0$ on S_2 $\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = 0$

Likewise, $y = 0$ on S_4 and $z = 0$ on S_6 both surface integrals are zero.
On S_3 $y = 1$ (and on S_5 $z = 1$)

$$\text{flux} = \iint_{S_3} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_3} z^2 y \, dydz = \int_{x=0}^{x=1} \int_{z=0}^{z=1} z^2 \, dxdz = 1/3$$

$$\text{flux} = \iint_{S_5} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_5} xz \, dxdy = \int_{x=0}^{x=1} \int_{y=0}^{y=1} x \, dxdy = 1/2$$

$\text{flux} = 1 + 1/3 + 1/2 = 11/6$ (same result)