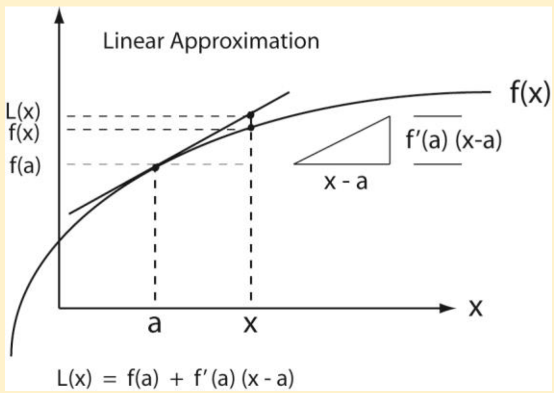
Linear Approximation of a Function with Two Independent Variables

In a Nut Shell: Recall that the "linear approximation", L(x), of a function of one independent variable, x, provides a way to find the value of f(x) at a neighboring point of a where x is a point close to a. This approximation directly relates to the slope of f(x) at x = a.

The figure below depicts the linear approximation, L(x), of the function, f(x).



where f(x) is the value of the function f at x
f(a) is the value of the function f(x) at x = a;
pick x to be slightly larger or slightly smaller than a
x - a is the interval; it is best to have a small interval
f '(a) is the slope of the function f(x) at x = a
L(x) is the linear approximation of f(x) near x = a

$$L(x) = f(a) + df(a)/dx \; (x - a)$$

One can extend the linear approximation of a function, f(x,y) of two independent variables say x and y by noting that the "surface", f(x,y), may change its slope in both the x and y-coordinate directions. In this case the relevant slopes at (x,y) = (a,b) are:

$\partial f(x,y)/\partial x$ evaluated at (x,y) = (a,b) i.e. change of slope in the x-direction

and

$\partial f(x,y)/\partial y$ evaluated at (x,y) = (a,b) i.e. change of slope in the y-direction

So the linear approximation of a function of two independent variables is:

$$L(x,y) = f(a,b) + \partial f(a,b)/\partial x \, [x - a] + \partial f(a,b)/\partial y \, [y - b]$$

Example: Find the linear approximation of the function, $f(x,y) = x^2e^y$ near the point (1,0). Use it to evaluate the function at (1.1, 0.1).

Recall:

$$\mathbf{L(x,y) = f(a,b) + \partial f(a,b)/\partial x\, [\, x - a\,] + \partial f(a,b)/\partial y\, [\, y - b\,]}$$

In this example $a = 1$, $b = 0$, $x = 1.1$, and $y = 0.1$.

First calculate the partial derivatives.

$\partial f/\partial x = 2x\, e^y$ and $\partial f/\partial y = x^2e^y$

Then evaluate them at (1,0).

$\partial f/\partial x|_{(1,0)} = 2$ and $\partial f/\partial|_{(1,0)} = 1$, Also $f(1,0) = 1$

The linear approximation becomes

$L(1.1, 0.1) = f(1,0) + \partial f/\partial x\, |_{(1,0)}\, [x - 1] + \partial f/\partial y\, |_{(1,0)})\, [y - 0]$

with $x = 1.1$ and $y = 0$.

The result is:

$L(1.1,0.1) = 1 + 2(1.1 - 1) + 1(0.1 - 0) = 1 + 0.3 = 1.3$