Change of Variables in Integrals

In a Nut Shell: Sometimes the evaluation of a double integral over a region, R, is difficult due to the complexity of the integrand, F(x,y), the shape of the region, R, or both.

$$I = \iint_R F(x,y)\, dx\, dy$$

In such cases, it may prove helpful to change the variables of integration from (x,y) to (u,v) to simplify either the integrand, the region of integration, or both. **Recall** change of variables (i.e. from rectangular to polar coordinates) was sometimes helpful in evaluating single integrals.

Strategy: Find a transformation, called the Jacobian transformation, $J_T(u,v)$, that simplifies the integrand, the region of integration, R, or both. This is the hard part. There is no set strategy. Two suggestions are to look at the integrand and pick a substitution that simplifies it or look at the region of integration and again pick a substitution that simplifies it.

Consider regions, R, in the xy-plane that are parallelograms. Let dx dy be the original element of area for region R and $J_T(u,v)$ du dv be the rectangular element of area of the transformed region, S, as shown in the figure below.

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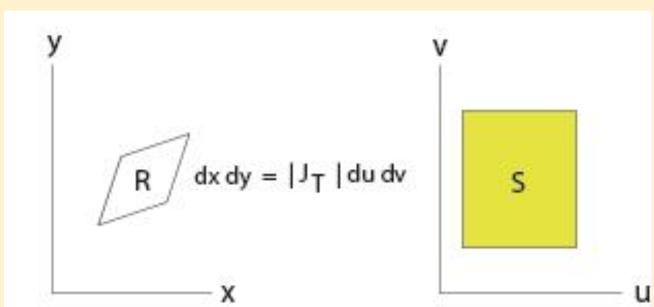


Then given sufficient continuity $dx\, dy = | J_T(u,v) |\, du\, dv$

where = $J_T(u,v)$ = the Jacobian transformation defined as follows:

$$J_T(u,v) = \det \begin{matrix} \partial x/\partial u & \partial x \partial v \\ \partial y/\partial u & \partial y \partial v \end{matrix}$$

where $| J_T(u,v) |$ is the magnitude of the Jacobian

and where det denotes the magnitude of the 2 x 2 determinant of the partial derivatives. The integral then becomes:

$$I = \iint_R F(x,y)\, dx\, dy = \iint_S G(u,v)\, | J_T(u,v) |\, du\, dv$$

R is the original region of integration and S is the transformed region of integration.

As mentioned the challenging part in changing variables from (x,y) to (u,v) is to find the transformation $x = x(u,v)$ and $y = y(u,v)$ from the region R in the xy-plane to region S in the uv-plane. Then use this information to calculate the Jacobian transformation, $J_T(u,v)$.

i.e. Use the Jacobian, $J_T(u,v)$, to rewrite the transformed integral as follows:

$$I = \iint_R F(x,y)\, dA = \iint_S G(u,v)\, |J_T(u,v)|\, du\, dv$$

The table below details one strategy to find the Jacobian transformation from parallelogram regions, R, in the xy-plane to rectangular regions, S, in the uv-plane.

Step 1 Graph the region, R, defined by:
$y - f_1(x) = C_1$, $y - f_1(x) = C_2$, by $y - f_2(x) = D_1$, $y - f_2(x) = D_2$.

Step 2 Determine the coordinates of the vertices for each corner of the region in R and the equations describing each "side" of region R.

Step 3 Find the new variables, u and v, by setting $u = y - f_1(x)$ and $v = y - f_2(x)$ where $C_1 \leq u \leq C_2$ and $D_1 \leq v \leq D_2$.

Step 4 The coordinates of the vertices for each corner of the region in S are (C_1, D_1), (C_1, D_2), (C_2, D_1), and (C_2, D_2) . Plot the rectangle, S.

Step 5 The limits of integration in the u-v plane are $C_1 \leq u \leq C_2$ and $D_1 \leq v \leq D_2$.

Step 6 Calculate the Jacobian, $J_T(u,v)$ by taking the partial derivatives $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, and $\partial y/\partial v$.

Step 7 Use the Jacobian to evaluate the integral, I,

$$I = \iint_R F(x,y)\, dA = \underset{S}{\int_{D_1}^{D_2}\int_{C_1}^{C_2}} G(u,v)\, |J_T(u,v)|\, du\, dv$$

Example: Use a change in variables to evaluate the integral

$$\iint_R (x + 2y)/[\cos(x-y)]\, dx\, dy \quad .$$

where R is a complicated region enclosed by the parallelogram in the x-y plane defined by intersection of the following lines as shown in the figure on the left below:

$$y = x, \quad y = x - 1 \qquad \text{and} \qquad x + 2y = 0, \quad x + 2y = 2$$

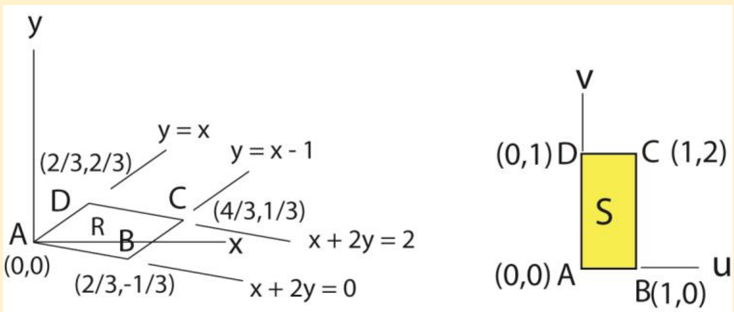
and $F(x,y) = (x + 2y)/[\cos(x-y)]$, is a complicated function

Strategy: Seek a transformation that will simplify the integrand, F(x,y), and also convert the parallelogram in R to a rectangular region, S in the u-v plane.
Rewrite the lines, boundaries of R, as follows:

$f_1(x) = x - y = 0$ and $f_2(x) = x + 2y = 0,$
$f_1(x) = x - y = 1$ and $f_2(x) = x + 2y = 2$

Now try the transformation coordinates $u = x - y$ and $v = x + 2y$

Note: $0 \leq u \leq 1$ and $0 \leq v \leq 2$ which bounds a rectangle, S, in the uv-plane.

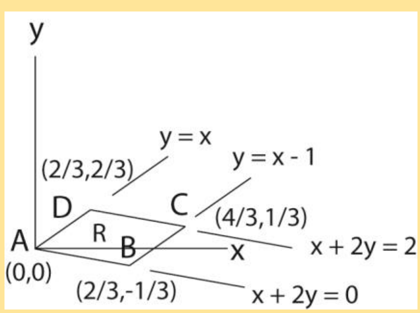


Note two simplifications occur with this transformation of coordinates.

1. The integrand changes from F(x,y) = = (x + 2y)/[cos(x-y)] to G(u,v) = v/cos (u).

2. The region, R, changes from a parallelogram to a rectangle S in the uv-plane.

From the graph of the region R (shown below) x - y = 0, x - y = 1, 2y + x = 0, 2y + x = 2



Try the transformation coordinates $u = x - y$ and $v = x + 2y$ so
$-1 \leq u \leq 0$ and $0 \leq v \leq 2$

and calculate the Jacobian transformation is: det means the 2 by 2 determinant

$$J_T(u,v) = \det \begin{matrix} \partial x/\partial u & \partial x \partial v \\ \partial y/\partial u & \partial y \partial v \end{matrix} = \partial(x,y)/\,\partial(u,v)$$

Method 1: Calculate the Jacobian by solving for x and y in terms of u and v and then calculate the partial derivatives $\partial x/\partial u$, $\partial x\partial v$, $\partial y/\partial u$, and $\partial y\partial v$. Here

$$v - u = 3y, \quad y = (1/3)(v - u) \quad \text{and} \quad x = u + y = (2/3)u + (1/3)v$$

So $\partial x/\partial u = 2/3$, $\partial x\partial v = 1/3$, $\partial y/\partial u = -1/3$, $\partial y\partial v = 1/3$

Therefore $J_T(u,v) = (2/3)(1/3) - (-1/3)(1/3) = 1/3$

Method 2: Recognize that the product of Jacobians is $[J_T(u,v)]\,[J_T(x,y)] = 1$

Then $J_T(u,v) = 1 / J_T(x,y)$ where

$$J_T(x,y) = \det \begin{matrix} \partial u/\partial x & \partial u\partial y \\ \partial v/\partial x & \partial v\partial y \end{matrix} = \partial(u,y)/\partial(x,y) = \det \begin{matrix} 1 & -1 \\ 1 & 2 \end{matrix} = 3$$

Therefore $J_T(u,v) = 1/3$. (same result)

$$I = \int\int F(x,y)\,dx\,dy = \int\int G(u,v)\,|J_T(u,v)|\,du\,dv = \int\int G(u,v)\,(1/3)\,du\,dv$$

Now use $J_T(u,v) = 1/3$ to transform integration from the xy-plane to the uv-plane.

$$I = \int\int F(x,y)\,dx\,dy = \int\int G(u,v)\,|J_T(u,v)|\,du\,dv = \int\int G(u,v)\,(1/3)\,du\,dv$$

Here $F(x,y) = (x + 2y)/[\cos(x-y)]$ and $u = x + y$, $v = x + 2y$ so the function $F(x,y)$ simplifies to

$$G(u,v) = v / \cos(u)$$

Thus, using the Jacobian transformation, the integral becomes

$$I = \int\int_R (x + 2y) / [\cos(x-y)]\,dx\,dy = \underset{S}{\int_{v=0}^{v=2} \int_{u=0}^{u=1}} [v / \cos u]\,[1/3]\,du dv$$

Change the order of integration.

$$I = \int_{u=0}^{u=1} \int_{v=0}^{v=2} [v \sec u][1/3]\,dv du = (2/3) \int_{u=0}^{u=1} \sec u\,du$$

$$I = (2/3) \ln|\sec u + \tan u| \Big|_0^1 = 2/3 \ln|\sec 1 + \tan 1| \qquad \text{(result)}$$

Note: In this example both the integrand, F(x,y) and the region of integration, R, were simplified using the transformation from (x,y) to (u,v) coordinates.

Next consider an example where the region, R, in the xy-plane is not a parallelogram. Seek a transformation that simplifies the integrand, yields a simpler region, S, in the uv-plane, or both.

Example: Use a change in variables to evaluate the integral

$$\iint_R ([x\,y]\,dA\ .$$

where the region R is a complicated region in the x-y plane defined by the intersection of the Lines $y = x$ and $y = 2x$ and also by the hyperbolas $xy = 1$ an d $xy = 3$.

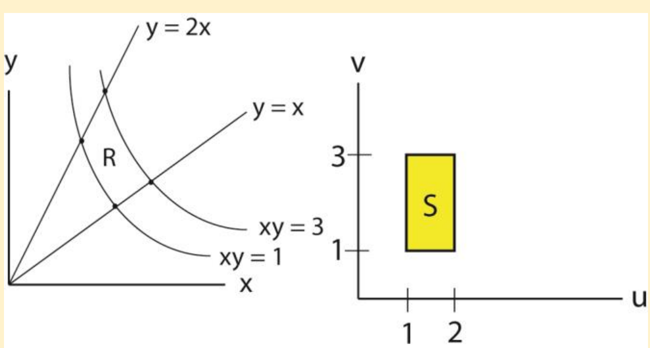
and $F(x,y) = xy$ See the figure below.

Strategy: Seek a transformation that will convert the region, R, to a rectangular region, S in u-v plane. Rewrite the lines, boundaries of R, as follows:

$f_1(x) = y/x = 1$ and $f_2(x) = xy = 1$
$f_1(x) = y/x = 2$ and $f_2(x) = xy = 3$

Now try the transformation coordinates $u = y/x$ and $v = xy$

Note: $1 \le u \le 2$ and $1 \le v \le 3$ which bounds a rectangle, S, in the uv-plane as shown in the figure below on the right.



Note two simplifications occur with this transformation of coordinates.

1. The integrand changes from $F(x,y) = xy$ to $G(u,v) = v$.

2. The region, R, changes from a complicated shape to a rectangle S in the uv-plane.

Try the transformation $u = y/x$ and $v = x y$

where $1 \le u \le 2$ and $1 \le v \le 3$

Then calculate the Jacobian transformation.

$$J_T(u,v) = \det \begin{matrix} \partial x/\partial u & \partial x \partial v \\ \partial y/\partial u & \partial y \partial v \end{matrix} = \partial(x,y)/\partial(u,v)$$

Method : Recognize that the product of Jacobians is $[J_T(u,v)]\,[J_T(x,y)] = 1$

Then $J_T(u,v) = 1 / J_T(x,y)$ where

$$J_T(x,y) = \det \begin{matrix} \partial u/\partial x & \partial u \partial y \\ \partial v/\partial x & \partial v \partial y \end{matrix} = \det \begin{matrix} -y/x^2 & 1/x \\ y & x \end{matrix} = -y/x - y/x$$

$$J_T(x,y) = -2\,y/x = -2u$$

Therefore $J_T(u,v) = -(1/2u)$

$$I = \int\int F(x,y)\,dx\,dy = \int\int G(u,v)\,|J_T(u,v)|\,du\,dv = \int\int G(u,v)\,(1/2u)\,du\,dv$$

But $G(u,v) = v$

So $$I = \int_{v=1}^{v=3} \int_{u=1}^{u=2} (v / 2u)\,du\,dv$$

So $$I = \int_{v=1}^{v=3} (v)\,(1/2) \ln u \Big|_{u=1}^{u=2})\,dv = (1/2) \ln 2 \int_{v=1}^{v=3} v\,dv$$

$$I = (1/2) \ln 2 \left[v^2/2 \right|_{v=1}^{v=3} = (1/4) \ln 2\,[\,3^2 - 1^2\,] = 2 \ln 2 \qquad \text{(result)}$$